

FREE DIVISORS IN PREHOMOGENEOUS VECTOR SPACES

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ABSTRACT. We study *linear free divisors*: free divisors arising as discriminants in prehomogeneous vector spaces, and in particular in quiver representation spaces. We show that the quivers for which this can occur have no (oriented or unoriented) cycles, and we prove some results on homogeneity and the logarithmic comparison theorem.

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1. INTRODUCTION

A reduced hypersurface $D \subset \mathbb{A}_{\mathbb{C}}^n = V$ is called a linear free divisor if $\text{Der}(-\log D)$ admits a global basis of logarithmic vector fields. It turns out that D is then the discriminant in a prehomogeneous vector space (G, ρ, V) with $\dim G = n$ defined by a homogeneous polynomial $f \in \mathbb{C}[V^*]$ of degree $\deg(f) = n$. In this sense,

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this class of divisors is opposite to the class of central hyperplane arrangements, where the irreducible components are linear but the basis of logarithmic vector fields is in general not. (Only the normal crossing divisor is a member of both classes.) Recently linear free divisors have been subject to intensive research [BM06, GMNRS09, dGMS08, Sev09].

The main two questions on linear free divisors to be considered in this article are also central questions in the case of hyperplane arrangements:

- (1) Which prehomogeneous vector spaces define linear free divisors?
- (2) Which linear free divisors satisfy the logarithmic comparison theorem?

We investigate these questions in general and in the (combinatorial) special case of linear free divisors that arise as discriminants in quiver representation spaces, so-called quiver linear free divisors.

Concerning question (1), we give a representation theoretic characterization of (quiver) linear free divisors in Theorems 2.1 and 3.4. Our main results in this direction, stated in Theorems 2.5 and 3.10, can be summarized as follows.

Theorem 1.1. (1) *For a linear free divisor with reductive group, the number of its irreducible components coincides with the number of irreducible summands of the corresponding prehomogeneous space.*

(2) *Quiver linear free divisors occur only in the representation spaces of quivers without (oriented or unoriented) cycles.*

In Section 2.3 we cover special cases for which we can give a complete description of all

- (1) irreducible linear free divisors with reductive group, or equivalently,
- (2) linear free divisors with reductive group and 1-dimensional center, and all
- (3) linear free divisors with Abelian group.

Our investigations concerning question (2) are motivated by the following result and conjecture. Denote by $j: U = V \setminus D \hookrightarrow V$ the inclusion of the complement. One says that the *logarithmic comparison theorem* holds for D if the de Rham morphism

$$\Omega_X^\bullet(\log D) \longrightarrow j_*\Omega_U^\bullet \simeq \mathbb{R}j_*\mathbb{C}_U$$

is a quasi-isomorphism. The name is motivated by Grothendieck's comparison theorem ([Gro66, Thm. 2]) which asserts that the de Rham morphism

$$\Omega_X^\bullet(*D) \longrightarrow j_*\Omega_U^\bullet \simeq \mathbb{R}j_*\mathbb{C}_U$$

is a quasi-isomorphism.

Following earlier work of Castro-Jiménez et al. in [CJGVHUU07], Narváez-Macarro [NM08, Rem. 1.7.4] proved the following sufficient condition for the logarithmic comparison theorem to hold.

Theorem 1.2. *Locally weakly quasihomogeneous free divisors satisfy the logarithmic comparison theorem.*

The following conjecture has been established for Koszul free divisors and for free divisors in dimension up to 3 [GS06, Cor. 1.5, Thm. 1.6].

Conjecture 1.3. *If a free divisor satisfies the logarithmic comparison theorem then it must be strongly Euler homogeneous.*

The homogeneity properties in Theorem 1.2 and Conjecture 1.3 are defined and characterized in Section 2.4. Our main results in this direction, stated in Theorems 3.17, 3.22, and 3.23, can be summarized as follows. We refer to a quiver linear free divisor as tame if it is defined by a tame quiver as defined in Section 3.7.

Theorem 1.4. *All quiver linear free divisors are strongly Euler homogeneous. They are locally weakly quasihomogeneous at points whose corresponding representation is not regular. All tame quiver linear free divisors are locally weakly quasihomogeneous.*

Using Theorem 1.2, we derive the following

Corollary 1.5. *The logarithmic comparison theorem holds for tame quiver linear free divisors.*

In Section 3.8, we study quiver linear free divisors with respect to the reflection functors. We summarize our main results on this subject, formulated in Theorem 3.31 and Corollary 3.32, in the following theorem, whose second statement is relevant only in case of a regular generic representation, see §3.9.

Theorem 1.6. *Reflection functors preserve the class of quiver linear free divisors. Modulo reflection functors and adding arrows with associated head and tail dimensions equal to 1, each equivalence class contains a divisor represented by an oriented bipartite graph.*

In Theorem 3.33, we give a simple formula for the degrees of the irreducible components of a quiver linear free divisor.

2. LINEAR FREE DIVISORS

2.1. Definition and characterization. Consider a reduced hypersurface $D \subset \mathbb{A}_{\mathbb{C}}^n = V$, and let $\mathcal{O} = \mathcal{O}_V$ be the sheaf of regular functions. We use coordinates $x = x_1, \dots, x_n$ on V and denote by $\partial = \partial_1, \dots, \partial_n$, where $\partial_i = \partial/\partial x_i$, the corresponding partial derivatives. One calls D a *linear free divisor* if the sheaf $\text{Der}(-\log D)$ of logarithmic vector fields along D (cf. [Sai80]) is freely generated by the germs of globally defined linear vector fields. By Saito's criterion [Sai80, Thm. 1.8.(ii)], such a D is the zero locus of the reduced homogeneous polynomial

$$(2.1) \quad f = \det(\delta_i(x_j)) \in \mathbb{C}[V^*]$$

of degree n , where $\delta_1, \dots, \delta_n$ is any basis of the Lie algebra

$$\mathfrak{l} := \Gamma(V, \mathcal{L})_0, \quad \mathcal{L} := \text{Der}(-\log D),$$

of global linear logarithmic vector fields. Conversely, for any n -dimensional Lie algebra of linear vector fields \mathfrak{l} , f defines a linear free divisor if it is reduced.

In more abstract terms, D is a linear free divisor if the inclusion $\mathfrak{l} \hookrightarrow \Gamma(V, \mathcal{L})$ gives rise to an isomorphism of free \mathcal{O} -modules

$$\mathfrak{l} \otimes_{\mathbb{C}} \mathcal{O} \cong \mathcal{L}.$$

Then $\bigwedge^n \mathfrak{l}$ and $\bigwedge^n \Gamma(V, \text{Der}_{\mathbb{C}} \mathcal{O})_{-1}$ are one dimensional subspaces of $\bigwedge^n \Gamma(V, \text{Der}_{\mathbb{C}} \mathcal{O})$ and $f \in \mathbb{C}[V^*]$ is defined by

$$f \cdot \bigwedge^n \Gamma(V, \text{Der}_{\mathbb{C}} \mathcal{O})_{-1} = \bigwedge^n \mathfrak{l}$$

up to a constant multiple.

In terms of algebraic group actions, linear freeness can be described as follows. For a general D , define G to be largest connected subgroup of $\mathrm{GL}(V)$ preserving D . Denote by $\rho: G \rightarrow \mathrm{GL}(V)$ the identity map. By [GMNRS09, Lem. 2.2] the infinitesimal action of G gives rise to an isomorphism

$$(2.2) \quad \omega_0: \mathfrak{g} \rightarrow \mathfrak{l}, \quad A \mapsto x^t A^t \partial$$

inducing a map

$$(2.3) \quad \omega: \mathfrak{g} \otimes \mathcal{O} \rightarrow \mathcal{L}.$$

So D is linear free if the latter map is an isomorphism. Brion gave the following characterization of linear freeness in the unpublished notes [Bri07b].

Theorem 2.1. *D is linear free if and only if the following conditions hold:*

- (1) $V \setminus D$ is a unique G -orbit, and the corresponding isotropy groups are finite.
- (2) The smooth part in D of each irreducible component of D is a unique G -orbit, and the corresponding isotropy groups are extensions of finite groups by the multiplicative group \mathbb{G}_m .

Proof. By reflexivity of \mathcal{L} and since $\mathrm{Sing} D$ has codimension 2, (2.3) being an isomorphism can be checked on $V \setminus \mathrm{Sing} D$. In the terminology of [Bri07a], this condition means that the pair $(V \setminus \mathrm{Sing} D, D \setminus \mathrm{Sing} D)$ is log parallelizable under G . So the “only if” statement follows from Proposition 2.1.2 in loc. cit. applied to the (smooth) logarithmic strata $S_0 = V \setminus D$ and the irreducible components S_1, \dots, S_k of $D \setminus \mathrm{Sing} D$. In our situation, this Proposition states that $(V \setminus \mathrm{Sing} D, D \setminus \mathrm{Sing} D)$ is log parallelizable under G along $S \in \{S_0, \dots, S_k\}$ exactly if S is a unique G -orbit and the differential $d\rho_x$ of the normal action ρ_x of G_x at $x \in S$ is an isomorphism. In case these equivalent conditions hold true, the proof yields for each $x \in S_j$, $j = 1, \dots, k$, an isomorphism of exact sequences

$$(2.4) \quad \begin{array}{ccccccc} 0 & \longrightarrow & \mathfrak{g}_x & \longrightarrow & \mathfrak{g} & \longrightarrow & \mathfrak{g}/\mathfrak{g}_x \longrightarrow 0 \\ & & \downarrow d\rho_x & & \downarrow \omega_x & & \downarrow \bar{\omega}_x \\ 0 & \longrightarrow & N_{S_j/V, x} & \longrightarrow & \mathcal{L}/\mathfrak{m}_x \mathcal{L} & \longrightarrow & T_x S_j \longrightarrow 0 \\ & & \parallel & & \parallel & & \parallel \\ & & \mathbb{C}x_1 \partial_1 & & \mathbb{C}x_1 \partial_1 \oplus \bigoplus_{i=2}^n \mathbb{C} \partial_i & & \bigoplus_{i=2}^n \mathbb{C} \partial_i \end{array}$$

where D is defined by x_1 in the completion $\hat{\mathcal{O}}_x$. The map $d\rho_x$ is the differential of the normal action and ω_x and $\bar{\omega}_x$ are induced by ω from (2.3). We shall get back to this diagram after the proof.

For the “if” statement one has to show that $d\rho_x$ is an isomorphism for $x \in V \setminus \mathrm{Sing}(D)$, using [Bri07a, Prop. 2.1.2] again. This is clear for $x \in S_0$ by assumption (1). For $x \in S_j$, $j = 1, \dots, k$, the normal action of the reductive subgroup $\mathbb{G}_m \subseteq G_x$ from assumption (2) is induced by an action on a line L transversal to D at x through some character. Since $L \cap S_0$ is dense in L , this character can not be trivial. \square

Consider a linear free $D \subset V$ defined by $f \in \mathbb{C}[V^*]$. Then (G, ρ, V) is a prehomogeneous vector space and f is a relative invariant polynomial associated to some

(non-trivial) character $\chi \in X^*(G)$ by

$$(2.5) \quad f(\rho(g)x) = \chi(g)f(x).$$

This equality can be differentiated to

$$(2.6) \quad \omega_0(A)(f) = \langle d\rho(A)x, \nabla f \rangle = d\chi(A)f.$$

Define $H = \mathrm{GL}(V)_f^\circ \subset G$ where $\mathrm{GL}(V)_f = \ker \chi$ is the isotropy group of f and denote by $\mathfrak{h} = \ker d\chi$ the Lie algebra of H . If H is reductive (resp. semisimple or Abelian), we call D reductive (resp. semisimple or Abelian).

Proposition 2.2. *Let D be a reductive linear free divisor in V . Then the G -orbits in Theorem 2.1.(2) are also H -orbits with finite isotropy groups. For $t \neq 0$, the level sets $D_t = \mathbb{Z}(f-t) \subset V \setminus D$ are closed H -orbits with finite isotropy groups.*

Proof. As in [GMNRS09, Lem. 2.2], one shows that $\omega_0(\mathfrak{h}) = \Gamma(V, \mathcal{H})_0$ where $\mathcal{H} = \mathrm{Der}(-\log f)$. Then ω from (2.3) induces an isomorphism

$$(2.7) \quad \omega: \mathfrak{h} \otimes \mathcal{O} \rightarrow \mathcal{H}$$

and hence $\omega_x(\mathfrak{h}) = \mathcal{H}/\mathfrak{m}_x\mathcal{L}$. In (2.4), we may assume that x_1 is the image of f . Then, for each $x \in S_j$, $j = 1, \dots, k$, $T_x(\rho(H)x) = \omega_x(\mathfrak{h}) = \bigoplus_{i=2}^n \mathbb{C}\partial_i = T_x S_j$ and hence $\mathfrak{h}_x = 0$. This proves the first statement.

The same argument can be applied to the (smooth) level sets $D_t = \mathbb{Z}(f-t)$ of f of level $t \in \mathbb{C}^*$. Here we conclude from the isomorphism (2.7) that $T_x(\rho(H)x) = T_x D_t$ is $(n-1)$ -dimensional for each $x \in D_t$. Thus $\rho(H)x = D_t$ as the latter is connected. The second statement follows. \square

2.2. The reductive case.

Proposition 2.3. *Let D be a reductive linear free divisor in V . Then each H -orbit in D contains 0 in its closure. In particular, $\{0\}$ is the only closed H -orbit in D , or G -orbit in V .*

Proof. For the action of H on D , any invariant homogeneous polynomial must be constant by Proposition 2.2. So D is the null cone and 0 is in the closure of each H -orbit, see [Dol03, §8.1]. \square

As the character χ is non-trivial it is surjective and hence $f: V \rightarrow \mathbb{C}$ is surjective. The following corollary is therefore a consequence of Proposition 2.3 and the quotient criterion in [Kra84, Satz II.3.4].

Corollary 2.4. *If D is a reductive linear free divisor then $f: V \rightarrow \mathbb{C}$ is an H -quotient with null cone D . In particular, $\mathbb{C}[V]^H = \mathbb{C}[f]$.*

Theorem 2.5. *For a reductive linear free divisor D with group G , the number of irreducible components of D equals the number of irreducible G -modules in V , and also equals the dimension of the center of G . In particular, D is irreducible if and only if H is semisimple.*

Proof. We assume that G is reductive, so $G = Z \cdot S$ is an almost direct product of the connected center Z and the derived group S which is semisimple. Each irreducible component $D_i = \overline{S_i}$, $i = 1, \dots, k$, of D is defined by a relative invariant polynomial $f_i \in \mathbb{C}[V^*]$ with an associated character χ_i as in (2.5). These characters form a basis of the group $X_0^*(G)$ of all characters corresponding to relative invariants ([SK77, §4, Lem. 4]). Let G_0 denote the finite isotropy group of (any point in)

$S_0 = V \setminus D$. Then $X_0^*(G)$ is the subgroup of $\chi \in X^*(G)$ with $\ker \chi \supseteq G_0$ (which is independent of the choice of the point in S_0 since all the corresponding isotropy groups are conjugate). For if p is in the open orbit $S_0 = V \setminus D$, the orbit map $G \rightarrow Gp = S_0$ identifies the G_p -invariant functions on G with functions on S_0 . So a character χ that is trivial on G_p gives rise to a function f on S_0 , which, as is easy to check, is then a relative invariant with character χ .

Taking the $\#G_0$ -th power, induces the second inclusion in the chain

$$X_0^*(G) \subseteq X^*(G) \hookrightarrow X_0^*(G),$$

which proves the second equality in

$$k = \operatorname{rk} X_0^*(G) = \operatorname{rk} X^*(G) = \operatorname{rk} X^*(Z) = \dim Z.$$

The third equality is proved similarly using finiteness of $Z \cap S$ and that $X^*(S) = 1$ by semisimplicity of S .

Now consider the decomposition of V into irreducible G -modules

$$(2.8) \quad V = V_1 \oplus \cdots \oplus V_\ell$$

and denote by $Z_i \cong \mathbb{G}_m$ the center of $\operatorname{GL}(V_i)$. Then Z acts by a character σ_i on each V_i and $S \subseteq \operatorname{SL}(V_1) \times \cdots \times \operatorname{SL}(V_\ell)$, since the latter is the derived group of $\operatorname{GL}(V_1) \times \cdots \times \operatorname{GL}(V_\ell)$. We obtain a homomorphism

$$(2.9) \quad Z/(Z \cap S) \cong G/S \hookrightarrow \frac{\operatorname{GL}(V_1) \times \cdots \times \operatorname{GL}(V_\ell)}{\operatorname{SL}(V_1) \times \cdots \times \operatorname{SL}(V_\ell)} \cong Z_1 \times \cdots \times Z_\ell$$

where $Z \cap S$ is finite. In particular, $k \leq \ell$, which is indeed a well known fact for more general reductive prehomogeneous vector spaces. To see the opposite inequality in our case, it suffices to show that $Z_i \cong 1 \times \cdots \times 1 \times Z_i \times 1 \times \cdots \times 1 \in \operatorname{GL}(V)$ lies in G for all $i = 1, \dots, \ell$, which makes (2.9) an isomorphism. Now $\mathfrak{l} \subset \mathfrak{gl}(V_1) \times \cdots \times \mathfrak{gl}(V_\ell)$, and so f , defined as in (2.1), is multihomogeneous with respect to the decomposition (2.8), and hence $Z_i \subseteq G$ as claimed. \square

2.3. Special cases. Based on Theorem 2.5 and the classification of irreducible prehomogeneous vector spaces by Sato and Kimura [SK77], we can give a complete description of all semisimple (i.e. irreducible reductive) linear free divisors.

In [SK77, §2], Sato and Kimura introduce the notion of *castling transformation*. Let $(\operatorname{SL}_n, \Lambda_1, V(n))$ denote the standard n -dimensional representation of SL_n . Two representations (G, ρ, V) and (G', ρ', V') are said to be castling transformations of one another if there exists a third representation $(\tilde{G}, \tilde{\rho}, V(m))$ such that

$$(2.10) \quad (G, \rho, V) \simeq (\tilde{G} \times \operatorname{SL}_n, \tilde{\rho} \otimes \Lambda_1, V(m) \otimes V(n))$$

and

$$(2.11) \quad (G', \rho', V') \simeq (\tilde{G} \times \operatorname{SL}_{m-n}, \tilde{\rho}^* \otimes \Lambda_1, V(m)^* \otimes V(m-n))$$

where $\tilde{\rho}^*$ is the contragredient representation of $\tilde{\rho}$ on the dual vector space $V(m)^*$ of $V(m)$. Two representations (G, ρ, V) and (G', ρ', V') are said to be in the same *castling class*, written $(G, \rho, V) \sim (G', \rho', V')$, if they are related by a finite number of castling transformations. In [SK77, §4, Prop. 18], it is shown that there is a one-to-one correspondence between relative invariants of representations related in this way, which, moreover, preserves the property of irreducibility. In addition, when (G, ρ, V) and (G', ρ', V') are related as in (2.10) and (2.11), and relative invariants f and f' correspond to one another, there is an integer d such that $\deg f = nd$ and $\deg f' = (m-n)d$.

We briefly review the proof from [SK77]. Identify $V(m) \otimes V(n)$ with the space $M(m, n)$ of $m \times n$ matrices, and $V(m)^* \otimes V(m-n)$ with $M(m, m-n)$. The representations $\tilde{\rho} \otimes \Lambda_1$ and $\tilde{\rho}^* \otimes \Lambda_1$ are defined by

$$(g_1, g_2) \cdot (x_{i,j}) = g_1(x_{i,j})g_2^t, \quad (g_1, g_2) \cdot (y_{i,j}) = (g_1^t)^{-1}(y_{i,j})g_2^t,$$

respectively. Let $\text{SI}(\tilde{\rho} \otimes \Lambda_1)$ and $\text{SI}(\tilde{\rho}^* \otimes \Lambda_1)$ be the algebras of polynomial relative invariant for $\tilde{\rho} \otimes \Lambda_1$ and for $\tilde{\rho}^* \otimes \Lambda_1$ respectively. Any member f of $\text{SI}(\tilde{\rho} \otimes \Lambda_1)$ must in particular be an absolute invariant for Λ_1 , and thus, by e.g. [Wey97, p. 45], must be a polynomial F in the $n \times n$ minors of $(x_{i,j})$. One should think of F as a polynomial in the homogeneous coordinate ring of the Plücker embedding of $\text{Gr}(n, m)$ in $\mathbb{P} \wedge^n \mathbb{C}^m$. Similarly, any member of $\text{SI}(\tilde{\rho}^* \otimes \Lambda_1)$ must be a polynomial G in the $(m-n) \times (m-n)$ minors of $(y_{i,j})$. For any sequence $I = i_1, \dots, i_n$ of integers in $\{1, \dots, m\}$, let X_{i_1, \dots, i_n} denote the determinant of the matrix formed by rows i_1, \dots, i_n of $(x_{i,j})$. Similarly, let $Y_{j_1, \dots, j_{m-n}}$ denote the determinant of rows j_1, \dots, j_{m-n} of $(y_{i,j})$. There is a natural signed bijection between these two sets of minors,

$$X_I \mapsto Y'_I := \text{sign}(I, I^c) Y_{I^c}$$

where $|I| = n$, I^c is the complement of I in $\{1, \dots, m\}$, and $\text{sign}(I, I^c)$ is the sign of the permutation I, I^c of $1, \dots, m$. This bijection gives rise to an algebra isomorphism

$$(2.12) \quad \Phi: \mathbb{C}[(X_I)_{|I|=n}] \rightarrow \mathbb{C}[(Y'_I)_{|I|=m-n}]$$

sending $F((X_I))$ to $F((Y'_I))$. It is well defined because the relations between the X_I are the same as the relations between the Y'_I : both are the Plücker relations, generating the ideals of the (isomorphic) embeddings of the Grassmannians $\text{Gr}(n, m)$ and $\text{Gr}(m-n, m)$ in $\mathbb{P} \wedge^n \mathbb{C}^m$ and in $\mathbb{P} \wedge^{m-n} \mathbb{C}^m$ respectively. Sato and Kimura show that Φ is \tilde{G} -equivariant, and from this, [SK77, §4, Prop. 18] follows.

The following proposition summarizes results explicitly and implicitly contained in [SK77].

Proposition 2.6. *Suppose that $(G, \rho, V) \sim (G', \rho', V')$. Then the following statements holds true:*

- (1) *The generic isotropy subgroups of (G, ρ, V) and (G', ρ', V') are isomorphic.*
- (2) *$\dim G - \dim V = \dim G' - \dim V'$.*
- (3) *If (G, ρ, V) is a prehomogeneous vector space then so is (G', ρ', V') .*
- (4) *If (G, ρ, V) is a prehomogeneous vector space and the complement of the open orbit is a linear free divisor with group G , then the same goes for (G', ρ', V') , mutatis mutandis. The number of irreducible components of these divisors are the same.*

Proof.

- (1) is proved in [SK77, §2, Prop. 9].
- (2) follows from the construction, and
- (3) follows from (1) and (2).

(4) Suppose that $\dim G = \dim V$ and (G, ρ, V) is prehomogeneous. Denote the complements of the open orbits in V and V' by D and D' . If D is a linear free divisor with group G , then $\dim G = \dim V = \deg D$, and so by (2), $\dim G' = \dim V'$. The defining equation of the discriminant in a prehomogeneous vector space is the product of generators of the semigroup of polynomial relative invariants. The isomorphism Φ of (2.12) induces an isomorphism between the respective semigroups

of polynomial relative invariants, multiplying each degree by $(m-n)/n$. This shows that also $\deg D' = \dim V'$. So D' is a linear free divisor with group G' . \square

In [SK77], Sato and Kimura classified irreducible prehomogeneous vector spaces up to castling transformations. We have seen in Theorem 2.5 that semisimple (i.e. irreducible reductive) linear free divisors live in irreducible representations of their group. Thus, up to castling transformations, every semisimple linear free divisor appears as the complement of the open orbit in one of the prehomogeneous vector spaces classified in [SK77]. We obtain

Theorem 2.7. *Up to castling transformations, there are only four semisimple (i.e. irreducible reductive) linear free divisors:*

- (1) $D = \{0\} \subset \mathbb{C}$ with $H = \{e\}$.
- (2) $D = \mathbb{Z}(y^2z^2 - 4xz^3 - 4y^3w + 18xyzw - 27w^2x^2) \subset \mathbb{C}^4$ with $H = \mathrm{SL}_2(\mathbb{C})$ acting naturally on $S^3\mathbb{C}^2 \cong \mathbb{C}^4$ (see [GMNRS09, Ex. 1.4.(2), Tab. 6.1]).
- (3) $D \subset \mathbb{C}^{12}$ with $H = \mathrm{SL}_3(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$ from [SK77, §5, Prop. 11].
- (4) $D \subset \mathbb{C}^{40}$ with $H = \mathrm{SL}_5(\mathbb{C}) \times \mathrm{SL}_4(\mathbb{C})$ from [SK77, §5, Prop. 15].

The next result describes the opposite extreme of Abelian linear free divisors.

Theorem 2.8. *The normal crossing divisors are the only Abelian linear free divisors.*

Proof. Let D be a linear free divisor. By [GMNRS09, Thm. 6.1], \mathfrak{t} has a basis $\sigma_1, \dots, \sigma_s, \nu_1, \dots, \nu_{n-s}$ where the σ_i are simultaneously diagonalizable and the ν_i are nilpotent. It is sufficient to assume only that $[\sigma_i, \nu_j] = 0$. Then there is a coordinate system x_1, \dots, x_n on V such that the σ_i are diagonal and the ν_i are block diagonal with respect to the largest blocks in which each σ_i has a single eigenvalue. In each such block, the ν_i can be made simultaneously strictly triangular. Assume that there is a block of size at least 2×2 . Then the defining equation f of D from (2.1) is the determinant of a matrix of the form

$$\begin{pmatrix} x_1 & x_2 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \\ 0 & 0 & \cdots \\ 0 & x_1 & \cdots \\ \vdots & \vdots & \\ 0 & x_1 & \cdots \end{pmatrix},$$

and hence divisible by x_1^2 . As D is reduced, all blocks have size 1×1 and hence $s = n$. But the $\sigma_1, \dots, \sigma_n$ are linearly independent, and hence we can assume that $\sigma_i = x_i \partial_{x_i}$ and then $f = x_1 \cdots x_n$ defines a normal crossing divisor D . \square

2.4. Euler homogeneity and weak quasi-homogeneity. A hypersurface D in V is *Euler homogeneous* at $p \in D$ if $f \in \mathfrak{m}_p \langle \frac{\partial f}{\partial x_i} \rangle$ for some, and hence any, local defining equation f of D at p . This condition can be rephrased by saying that $\chi(f) = f$ for some vector field $\chi \in \mathrm{Der}(-\log D)_p$ with $\chi(p) = 0$. Such a vector field is called an Euler vector field at p . If the preceding condition holds for all $p \in D$, then D is called *strongly Euler homogeneous*.

Proposition 2.9. *A linear free divisor D is Euler homogeneous at $p \in D$ if and only if $G_p^\circ \not\subseteq H$.*

Proof. For a linear free divisor D , Euler homogeneity at p is equivalent to $\omega_0(\text{id})(p) \in \omega_0(\mathfrak{h})(p)$ [GMNRS09, Lem. 7.5]. This means that $\omega_0(\text{id})(p) = \omega_0(A)(p)$ for some $A \in \mathfrak{h}$ and hence $\omega_0(\text{id} - A)(p) = 0$ or $\text{id} - A \in \mathfrak{g}_p$. As $\mathfrak{g} = \mathbb{C}\text{id} \times \mathfrak{h}$, this is equivalent to $\mathfrak{g}_p \not\subseteq \mathfrak{h}$. The claim follows. \square

Remark 2.10. The “if” part of Proposition 2.9 holds more generally for a prehomogeneous vector space (G, ρ, V) , a relative invariant $f \in \mathbb{C}[V]$ with associated character χ , $H = \ker \chi$, and $D = Z(f)$.

An algebraic hypersurface $D \subset V$ is *weakly quasihomogeneous* at $p \in D$, if $f = (x_1, \dots, x_n)A(x)(\frac{\partial f}{\partial x_i})^t \in \mathfrak{m}\langle \frac{\partial f}{\partial x_i} \rangle$ with $A(0)$ having only non negative eigenvalues for some, and hence any, local defining equation f of D at p . We notice this condition implies Euler homogeneity and that, by [GS06, Lem. 2.4], $A(0)$ has at least one positive eigenvalue.

By [Sch07], weak quasi-homogeneity means that, after a suitable analytic coordinate change, D is defined at p by a quasihomogeneous polynomial with respect to non-negative weights. If the preceding condition holds for all $p \in D$, then D is called locally weakly quasihomogeneous. If the attribute “weakly” is dropped, this means that all weights have to be strictly positive.

We call a representation $\rho: G \rightarrow \text{GL}(V)$ *positive* if the convex hull of its weights does not contain 0, and *non-negative* if the interior of the convex hull of its weights does not contain 0.

Proposition 2.11. *Let D be a linear free divisor in V . Then D is quasihomogeneous (weakly quasihomogeneous) at p if the normal representation $\rho_p: G_p \rightarrow \text{GL}(T_p V/T_p(Gp))$ is positive (non-negative).*

Proof. Assume that ρ_p is positive and set $N_p = T_p V/T_p(Gp)$. Then there is a 1-PSG $\lambda: \mathbb{G}_m \rightarrow G_p \rightarrow \text{GL}(N_p)$ that is positive on the weights of ρ_p . As $\mathbb{G}_m \subset G_p$ is reductive and preserves both $T_p V$ and $T_p(Gp)$, there is an affine linear transversal $S_p \subset V$ to the orbit Gp at p such that λ restricts to $\lambda': \mathbb{G}_m \rightarrow \text{GL}(S_p)$. Then $S_p \rightarrow N_p$ is an isomorphism and hence λ' is positive on its weights. This means that λ' defines an Euler vector field at p with strictly positive weights for $D' = D \cap S$. But locally analytically $(D, p) = \left(\mathbb{A}_{\mathbb{C}}^{\dim(Gp)}, 0 \right) \times (D', p)$ and hence with D' also D is quasihomogeneous at p . This statement about weak quasihomogeneity is proved analogously. \square

3. QUIVER REPRESENTATIONS

3.1. The category of quiver representations. A *quiver* (or directed graph) (Q_0, Q_1, s, t) consists of a finite set Q_0 of vertices, a finite set Q_1 of arrows, and source and target maps $s, t: Q_1 \rightarrow Q_0$. Without loss of generality, we shall always assume that Q is a connected graph. A Q -representation $M = (V_i, f_\alpha)$ consists of a family of finite \mathbb{C} -vector spaces V_i , $i \in Q_0$, and a family of homomorphisms $V_\alpha \in \text{Hom}(V_{s\alpha}, V_{t\alpha})$, $\alpha \in Q_1$. Homomorphisms $\phi = (\phi_i)_{i \in Q_0}: M \rightarrow N$, $\phi_i \in \text{Hom}(V_i, W_i)$, of two Q -representations $M = (V_i, f_\alpha)$ and $N = (W_i, g_\alpha)$ are defined by the obvious commutativity condition $f_\alpha \circ \phi_{s\alpha} = \phi_{t\alpha} \circ g_\alpha$.

To a Q -representation $M = (V_i, f_\alpha)$ one can associate the vector space $V = \bigoplus_{i \in Q_0} V_i$, maps $f_\alpha: V \rightarrow V$, $\alpha \in Q_1$, induced by f_α and the projection maps

$f_i: V \rightarrow V_i \hookrightarrow V$, $i \in Q_0$. These maps satisfy the relations $f_i^2 = f_i$ and $f_\alpha f_{s\alpha} = f_\alpha = f_{t\alpha} f_\alpha$ and all other products are zero. The path algebra $\mathbb{C}Q$ of Q is the associative algebra on the symbols e_i , $i \in Q_0$, and e_α , $\alpha \in Q_1$, subject to the preceding relations. It is finite \mathbb{C} -dimensional if and only if Q has no oriented loops. Note that

$$(3.1) \quad 1 = \sum_{i \in Q_0} e_i$$

is a decomposition into orthogonal idempotents. Clearly V becomes a $\mathbb{C}Q$ -module by $e_i \mapsto f_i$ and $e_\alpha \mapsto f_\alpha$.

This construction gives rise to the following equivalence [Bri08, Prop. 1.2.2].

Proposition 3.1. *The category of Q -representations is equivalent to that of left $\mathbb{C}Q$ -modules.*

This is the starting point for the study of quiver representations based on general module theory over an associative finite \mathbb{C} -dimensional algebra. It follows from the Fitting Decomposition that M is an indecomposable $\mathbb{C}Q$ -module if and only if $\text{End}_{\mathbb{C}Q} M = \mathbb{C} \text{id}_M \oplus I$ where I is a nilpotent ideal [Bri08, Lem. 1.3.3]. From this one deduces the following result [Bri08, Thm. 1.3.4].

Theorem 3.2. *Let M be a finite \mathbb{C} -dimensional $\mathbb{C}Q$ -module and $M = \bigoplus_{i=1}^s r_i M_i$ its decomposition into indecomposables M_i . Then $\text{End}_{\mathbb{C}Q} M = B \oplus I$ where $B \cong \prod_{i=1}^s \text{Mat}_{r_i \times r_i}(\mathbb{C})$ and I is a nilpotent ideal.*

The automorphism group $\text{Aut}_Q(M)$ of a Q -representation M is affine open in its Lie algebra $\text{End}_Q(M)$ and hence connected. This yields the following description of $\text{Aut}_Q(M)$ [Bri08, Prop. 2.2.1].

Proposition 3.3. *Let M be a Q -representation with decomposition into indecomposables $M = \bigoplus_{i=1}^s r_i M_i$. Then $\text{Aut}_Q(M) \cong U \rtimes \prod_{i=1}^s \text{GL}_{r_i}(\mathbb{C})$ where U is a closed normal unipotent subgroup.*

By (3.1), the left $\mathbb{C}Q$ -modules $P(i) = \mathbb{C}Qe_i$ are projective indecomposables and each left $\mathbb{C}Q$ -module M has a standard projective resolution

$$0 \longrightarrow \bigoplus_{\alpha \in Q_1} P(t\alpha) \otimes e_{s\alpha} M \xrightarrow{u} \bigoplus_{i \in Q_0} P(i) \otimes e_i M \xrightarrow{v} M \longrightarrow 0$$

where $u(a \otimes m) = a\alpha \otimes m - a \otimes \alpha m$ on the α -summand and v is the canonical multiplication map [Bri08, Prop. 1.4.1]. Using the isomorphism $\text{Hom}_{\mathbb{C}Q}(P(e), M) \cong eM$ defined by evaluation at the idempotent e , one obtains the Ringel exact sequence [Bri08, Cor. 1.4.2]

$$(3.2) \quad 0 \longrightarrow \text{Hom}_Q(M, N) \longrightarrow \bigoplus_{i \in Q_0} \text{Hom}(V_i, W_i) \xrightarrow{c_{M,N}} \bigoplus_{\alpha \in Q_1} \text{Hom}(V_{s\alpha}, W_{t\alpha}) \longrightarrow \text{Ext}_Q(M, N) \longrightarrow 0$$

associated to two Q -representations $M = (V_i, f_\alpha)$ and $N = (W_i, g_\alpha)$ where

$$c_{M,N}((\phi_i)_{i \in Q_0}) = (\phi_{t\alpha} f_\alpha - g_\alpha \phi_{s\alpha})_{\alpha \in Q_1}.$$

For $M = N$ it induces an exact sequence

$$(3.3) \quad 0 \longrightarrow \text{End}_Q(M) \longrightarrow \bigoplus_{i \in Q_0} \text{End}(V_i) \begin{array}{c} \curvearrowright \\ \xrightarrow{c_M} \bigoplus_{\alpha \in Q_1} \text{Hom}(V_{s\alpha}, V_{t\alpha}) \longrightarrow \text{Ext}_Q(M) \longrightarrow 0 \end{array}$$

where $c_M = c_{M,M}$ and $\text{Ext}_Q(M) = \text{Ext}_Q(M, M)$ denotes the self-extensions of M .

The Euler form on \mathbb{Q}^{Q_0} is defined by

$$\langle m, n \rangle_Q = \sum_{i \in Q_0} m_i n_i - \sum_{\alpha \in Q_1} m_{s\alpha} n_{t\alpha} = \langle m, En \rangle$$

where E is the Euler matrix defined by

$$E = I - A, \quad A = (\#\{\alpha \in Q_1 \mid s\alpha = i, t\alpha = j\})_{i,j}.$$

The Tits form is the associated quadratic form

$$(3.4) \quad q_Q(n) = \langle n, n \rangle_Q$$

and the Cartan matrix $C = E + E^t$ defines the associated symmetric bilinear form

$$\langle m, n \rangle_Q = \langle m, n \rangle_Q + \langle n, m \rangle_Q = \langle m, Cn \rangle.$$

For $m = \dim(M)$ and $n = \dim(N)$ it follows from (3.2) and (3.3) that

$$(3.5) \quad \begin{aligned} \langle m, n \rangle_Q &= \dim \text{Hom}_Q(M, N) - \dim \text{Ext}_Q(M, N), \\ q_Q(m) &= \dim \text{End}_Q(M) - \dim \text{Ext}_Q(M). \end{aligned}$$

By abuse of notation, we shall write $\langle M, N \rangle_Q$ for $\langle \dim(M), \dim(N) \rangle_Q$ and similarly for q_Q . For $M = (V_i, f_\alpha)$, we shall sometimes abbreviate $M(i) = V_i$, $M(\alpha) = f_\alpha$, $sM = \bigoplus_{\alpha \in Q_1} V_{s\alpha}$, and $tM = \bigoplus_{\alpha \in Q_1} V_{t\alpha}$.

3.2. Quiver representation spaces. Choosing bases of the V_i , each Q -representation V with fixed dimension vector $d = (\dim V_i)_{i \in Q_0}$ can be considered as a point in the representation space

$$(3.6) \quad \text{Rep}(Q, d) = \bigoplus_{i \rightarrow j \in Q_1} \text{Hom}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j}) = \bigoplus_{i \rightarrow j \in Q_1} \text{Mat}_{d_j \times d_i}(\mathbb{C})$$

on which the group $\text{GL}(Q, d) = \prod_{i \in Q_0} \text{GL}_{d_i}(\mathbb{C})$ acts by

$$(g_i)_{i \in Q_0} (x_\alpha)_{\alpha \in Q_1} = (g_{t\alpha} x_\alpha g_{s\alpha}^{-1})_{\alpha \in Q_1}.$$

In this way the $\text{GL}(Q, d)$ -orbits in $\text{Rep}(Q, d)$ correspond to the isomorphism classes of Q -representations with dimension vector d . Note that the central subgroup $\mathbb{C}^* \text{id} \subset \text{GL}(Q, d)$ acts trivially on $\text{Rep}(Q, d)$, so $\text{GL}(Q, d)$ acts via $\text{PGL}(Q, d) = \text{GL}(Q, d) / \mathbb{C}^* \text{id}$, and that, by the connectedness of Q , $\text{PGL}(Q, d) \hookrightarrow \text{GL}(\text{Rep}(Q, d))$. The isotropy group of $M \in \text{Rep}(Q, d)$ for the actions of $\text{GL}(Q, d)$ and $\text{PGL}(Q, d)$ can be identified with $\text{Aut}_Q(M)$ and $\text{Aut}_Q(M) / (\mathbb{C}^* \text{id}_M)$.

In the the sequence (3.3), c_M can be identified with the differential at id of the orbit map $\text{GL}(Q, d) \ni g \mapsto g \cdot M \in \text{Rep}(Q, d)$. This identifies $\text{End}_Q(M)$ with the isotropy Lie algebra of M in $\mathfrak{gl}(Q, d)$ and $\text{Ext}_Q(M)$ with the normal space in $\text{Rep}(Q, d)$ to the orbit $\text{GL}(Q, d)M$ at M , that is,

$$(3.7) \quad \text{Ext}_Q(M) = \text{Rep}(Q, d) / T_M(\text{GL}(Q, d)M)$$

Note that c corresponds to ω_0 in (2.2).

The following consequences of Theorem 2.1 and Proposition 3.3 are inspired by [Bri07b]. Let d be a sincere dimension vector on a quiver Q , that is, $d_i \geq 1$ for all $i \in Q_0$. We shall say that (Q, d) defines a linear free divisor D , and call D a *quiver linear free divisor*, if the discriminant of the action of $\mathrm{GL}(Q, d)$ on $\mathrm{Rep}(Q, d)$ is a linear free divisor D with associated group

$$G := \mathrm{PGL}(Q, d)$$

as in §2.1. In particular, such a D is a reductive linear free divisor. A Q -representation M is called a *brick* if $\mathrm{End}_Q(M) = \mathbb{C} \mathrm{id}_M$ and d is called a *Schur root* if $\mathrm{Rep}(Q, d)$ contains a brick.

Theorem 3.4. *(Q, d) defines a linear free divisor if and only if the following conditions are fulfilled:*

- (1) $q_Q(d) = 1$
- (2) d is a Schur root.
- (3) $\dim_{\mathbb{C}} \mathrm{End}_Q(N) = 2$ for any minimal degeneration $N \in \mathrm{Rep}(Q, d)$.

Condition (3) can be replaced by one of the following:

- (3') $\dim_{\mathbb{C}} \mathrm{Ext}_Q(N) = 1$ for any minimal degeneration $N \in \mathrm{Rep}(Q, d)$.
- (3'') All minimal degenerations lie in orbits of codimension 1.

Proof. Recall that $\mathrm{Aut}_Q(M)$ is connected and that $\mathrm{Aut}_Q(M)/(\mathbb{C}^* \mathrm{id}_M)$ identifies with the isotropy group G_M of $M \in \mathrm{Rep}(Q, d)$. By the following Remark 3.5, we can assume N in (3) to be decomposable. Then, by Proposition 3.3, (3) is equivalent to $\mathrm{Aut}_Q(M) \cong \mathbb{G}_m^2$.

Assume that (Q, d) defines a linear free divisor. Then, for $M \in V \setminus D$, the second condition in Theorem 2.1.(1) becomes $\mathrm{Aut}_Q(M) = \mathbb{C}^* \mathrm{id}_M$, and hence M is a brick and (2) holds true. The first condition in Theorem 2.1.(1) implies

$$\begin{aligned} q_Q(d) &= \dim \mathrm{GL}(Q, d) - \dim \mathrm{Rep}(Q, d) \\ &= \dim \mathrm{GL}(Q, d) - \dim(\mathrm{GL}(Q, d)M) = \dim \mathrm{Aut}_Q(M) \end{aligned}$$

and hence (1). Similarly, the conditions in Theorem 2.1.(2) imply (3) and (1).

By reversing these equalities, (1) and (2) show that a brick M has an orbit of dimension $\dim(\mathrm{GL}(Q, d)M) = \dim \mathrm{Rep}(Q, d)$, which is then open. Note that, in an irreducible variety, open and dense are equivalent properties of orbits and that such orbits are unique. Then (1) and (3) force its complement D to be a hypersurface and imply the conditions in Theorem 2.1.(2).

The last statement follows from (3.3). \square

Remark 3.5. By Kac's theorem [KR86, Thm. 4.3], conditions (1) and (2) in Theorem 3.4 can be rephrased by saying that d is a real Schur root and in this case there is a unique orbit of indecomposables. By (3.3) and Theorem 3.2, this orbit is open and consists of bricks.

Corollary 3.6. *Let (Q, d) define a linear free divisor D . Then the following statements hold:*

- (1) Up to isomorphism, there is a unique indecomposable Q -representation M with $\dim M = d$. Moreover, M is a brick in the open orbit $\mathrm{Rep}(Q, d) \setminus D$.
- (2) Any minimal degeneration N is the direct sum $N_1 \oplus N_2$ of two non-isomorphic bricks.

Proof.

- (1) This follows from Remark 3.5.
 (2) This follows from Theorem 3.2 and Theorem 3.4.(3).

□

The following result refines the statement in Corollary 3.6.(2).

Proposition 3.7. *Let (Q, d) define a linear free divisor D , and let $N \in \text{Rep}(Q, d)$ be a minimal degeneration. Then $N = N_1 \oplus N_2$ for two uniquely determined rigid submodules N_1 and N_2 whose dimension vectors d_1 and d_2 are real Schur roots. After perhaps permuting N_1 and N_2 , they satisfy the following conditions:*

$$(3.8) \quad \text{End}_Q(N_i) = \mathbb{C} \text{id}_{N_i}, \quad \text{Hom}_Q(N_1, N_2) = 0 = \text{Hom}_Q(N_2, N_1),$$

$$(3.9) \quad \text{Ext}_Q(N_i) = 0, \quad \text{Ext}_Q(N_1, N_2) = 0, \quad \text{Ext}_Q(N_2, N_1) = \mathbb{C}.$$

Proof. By Corollary (3.6).(2), we have $N = N_1 \oplus N_2$ satisfying the conditions in (3.8). The representations N_1 and N_2 are unique because $\text{Hom}_Q(N_i, N) = \text{Hom}_Q(N_i, N_1) \oplus \text{Hom}_Q(N_i, N_2)$ is 1-dimensional for $i \in \{1, 2\}$, and therefore must be generated by the canonical inclusion $N_i \hookrightarrow N_1 \oplus N_2 = N$.

The representation N_i is rigid for $i \in \{1, 2\}$ because for any representation $N'_i \in \text{Rep}(Q, d_i)$ in a sufficiently small neighborhood of N_i , the representation $N'_i \oplus N_j$, where $\{i, j\} = \{1, 2\}$, is in D by Corollary (3.6).(2) and therefore in the orbit of N , since this orbit is open in D . Hence by unicity, N_i and N'_i are isomorphic.

By (3.7), $\text{Ext}_Q(N) = \bigoplus_{i,j=1}^2 \text{Ext}_Q(N_i, N_j)$ is 1-dimensional, and $\text{Ext}_Q(N_i, N_i) = 0$ for $i \in \{1, 2\}$ by the rigidity of N_i . Therefore the conditions in (3.9) are satisfied after reordering N_1 and N_2 . Now (3.5) shows that $q_Q(N_i) = 1$ for $i \in \{1, 2\}$ and by rigidity this implies that d_1 and d_2 are real Schur roots. □

The representation N in Proposition 3.7 is, of course, a split extension of N_2 by N_1 . Where can we find a non-split extension? The answer is that every representation not in D is the total space of such an extension. In fact, using Ringel's construction we can exhibit the line L referred to in the proof of Theorem 2.1, that is, a G_N -invariant line L meeting D transversely at N , such that each point of $L \setminus D$ is a non-split extension of N_2 by N_1 .

Choose $\theta \in \text{Hom}_{\mathbb{C}}(sN_2, tN_1)$ so that its image under the inclusion $\text{Hom}_{\mathbb{C}}(sN_2, tN_1) \rightarrow \text{Hom}_{\mathbb{C}}(tN, hN)$, generates $\text{coker } c_{N,N}$. Construct a new representation $N(\lambda\theta)$ of Q , depending on the complex parameter λ , as an extension of N_2 by N_1 :

$$(3.10) \quad \begin{array}{ccccccc} 0 & \longrightarrow & N_1(s\alpha) & \longrightarrow & N_1(s\alpha) \oplus N_2(s\alpha) & \longrightarrow & N_2(s\alpha) \longrightarrow 0 \\ & & \downarrow N_1(\alpha) & & \downarrow \begin{pmatrix} N_1(\alpha) & \lambda\theta \\ 0 & N_2(\alpha) \end{pmatrix} & & \downarrow N_2(\alpha) \\ 0 & \longrightarrow & N_1(t\alpha) & \longrightarrow & N_1(t\alpha) \oplus N_2(t\alpha) & \longrightarrow & N_2(t\alpha) \longrightarrow 0 \end{array}$$

The representation in the center is $N(\lambda\theta)$. Evidently $N(\lambda\theta) = N$ when $\lambda = 0$. Since the tangent space to the line $L := \{N(\lambda\theta) : \lambda \in \mathbb{C}\}$ is spanned by θ , which does not belong to $\text{im}(c_{N,N}) = T_N D$, L is a complement to $T_N D$. It follows that except for a finite number of values of λ , $N(\lambda\theta) \notin D$ and the extension (3.10) is not split, and therefore generates $\text{Ext}_Q^1(N_2, N_1)$.

Lemma 3.8. *The line $L = \{N(\lambda\theta) : \lambda \in \mathbb{C}\}$ is invariant under the action of the isotropy group $G_N \subset \mathrm{GL}_{Q,d}$ of N .*

Proof. Clearly $G_N \subset \mathrm{Hom}_Q(N, N)$ so by what we have observed above, $G_N \subset \mathrm{Hom}_Q(N_1, N_1) \oplus \mathrm{Hom}_Q(N_2, N_2)$. Thus G_N is isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$ acting by scalar multiplication on each of the two summands. Let $(u, v) \in G_N$. The diagram

$$\begin{array}{ccc} N_1(s\alpha) \oplus N(s\alpha) & \xrightarrow{\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}} & N_1(s\alpha) \oplus N_2(s\alpha) \\ \begin{pmatrix} N_1(\alpha) & \lambda\theta \\ 0 & N_2(\alpha) \end{pmatrix} \downarrow & & \downarrow \begin{pmatrix} N_1(\alpha) & u^{-1}v\lambda\theta \\ 0 & N_2(\alpha) \end{pmatrix} \\ N_1(t\alpha) \oplus N_2(t\alpha) & \xrightarrow{\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}} & N_1(t\alpha) \oplus N_2(t\alpha) \end{array}$$

commutes; it follows that $(u, v) \cdot V(\lambda\theta) = V(u^{-1}v\lambda\theta)$. Thus G_{N_0} acts on L . \square

This construction gives a simple proof of Theorem 2.1 in the case of quiver representations.

Lemma 3.9. *Let d be a real Schur root of the quiver Q , and let $D \subset \mathrm{Rep}(Q, d)$ be the complement of the open orbit. If there is a $\mathrm{GL}(Q, d)$ -orbit which is open in the irreducible component D_i of D , then the equation f of the discriminant D is reduced along D_i .*

Proof. We continue to use the notation and assumptions of Lemma 3.9. Regard λ as a coordinate on ℓ . Identify G_N with $\mathbb{C}^* \times \mathbb{C}^*$ as described above, and consider the curve $\sigma(u) = (1, u) \in G_N$. As $(1, u) \cdot N(\lambda\theta) = N(u\lambda\theta)$, we have

$$(3.11) \quad \frac{d}{du} (\sigma(u) \cdot N(\lambda\theta))|_{u=1} = N(\lambda\theta).$$

Now consider $\sigma'(1)$ as an element of $\mathfrak{gl}(Q, d)$ via the inclusion $G_N \subset \mathrm{GL}(Q, d)$. Then (3.11) means that the linear vector field δ on $\mathrm{Rep}(Q, d)$ coming, via the infinitesimal action of $\mathrm{GL}(Q, d)$, from the tangent vector $\sigma'(1)$, restricts to $\lambda\partial_\lambda$ on ℓ .

Since the corank at N of the map $c_{N,N}$ is one, its image generates the tangent space $T_N D$. Let us choose linear vector fields $\delta_1, \dots, \delta_{r-1}$, where $r = \dim \mathrm{Rep}(Q, d)$, whose values at N generates this space. The determinant $\det(\delta, \delta_1, \dots, \delta_{r-1})$ is then reduced at N since it is a non-zero multiple of λ when restricted to ℓ . Since we know also that it is a multiple of the equation of the discriminant we conclude that the germ at N of this equation is reduced. \square

Theorem 2.5 gives the following strong restriction on those quivers Q which can define a linear free divisor.

Theorem 3.10. *If (Q, d) defines a linear free divisor then Q is a tree.*

Proof. Each summand of $\mathrm{Rep}(Q, d)$ in (3.6) is an irreducible $\mathrm{GL}(Q, d)$ -module. So their number equals $\ell = \#Q_1$. The number k of irreducible components of D equals $\mathrm{rk} X^*(\mathrm{PGL}(Q, d)) = \mathrm{rk} X^*(\mathrm{GL}(Q, d)) - 1 = \#Q_0 - 1$. By Theorem 2.5, we have $\#Q_1 = \ell = k = \#Q_0 - 1$ and hence Q is a tree, as it is connected. \square

Corollary 3.11. *Let (Q, d) define a linear free divisor. Then there is a partition $Q_0 = \bigcup_{k=0}^l Q_0^k$ such that $Q_1 \subset \bigcup_{k=1}^l Q_0^{k-1} \times Q_0^k$.* \square

3.3. Dynkin quivers. A quiver Q is a *Dynkin quiver* if its underlying graph is a Dynkin diagram of type A_n , D_n , E_6 , E_7 or E_8 . Gabriel proved in [Gab80] that the Dynkin quivers are precisely those of “finite representation type”: that is, such that for any d , $\text{Rep}(Q, d)$ contains only finitely many orbits. If Q is a Dynkin quiver and d is any dimension vector then $\text{Rep}(Q, d)$ therefore contains an open orbit. If moreover $q_Q(d) = 1$, then $\dim \text{PGL}(Q, d) = \dim \text{Rep}(Q, d)$; it follows that the Lie algebra of linear vector fields coming from the infinitesimal action of $\text{GL}(Q, d)$ on $\text{Rep}(Q, d)$ has dimension equal to that of $\text{Rep}(Q, d)$. Thus all that is needed for the discriminant D to be a linear free divisor is that the determinant of the Saito matrix should be reduced. Again, because Q is a Dynkin quiver, in each irreducible component of D there is an open orbit. It follows by Lemma 3.9 that the discriminant is reduced. This gives a concise proof of the following theorem from [BM06].

Theorem 3.12. *If Q is a Dynkin quiver and $q_Q(d) = 1$ then d is a real Schur root and the discriminant in $\text{Rep}(Q, d)$ is a linear free divisor.*

3.4. Relative invariants. For a prehomogeneous vector space (G, ρ, V) with singular set S , relative invariants are in correspondence with characters as in (2.5), and algebraic dependence corresponds to dependence of characters. The group of characters corresponding to relative invariants is the free Abelian group $X_0^*(G) = \{\chi \in X^*(G) \mid \chi(G_p) = 1, p \in V \setminus S\}$. The ring of polynomial relative invariants is a polynomial ring, and a basis is given by the defining equations of the irreducible components of S of codimension 1 [SK77, §4].

For a quiver Q and a real Schur root $d \in \mathbb{N}^{Q_0}$, $\text{Rep}(Q, d)$ is prehomogeneous for the action of $\text{GL}(Q, d)$ by Remark 3.5. Denote by $M(Q, d)^\perp$ the right orthogonal category of the generic representation $M(Q, d)$ in $\text{Rep}(Q, d)$, that is, the full subcategory of all M for which

$$\text{Hom}(M(Q, d), M) = 0 = \text{Ext}_Q(M(Q, d), M).$$

For $M \in M(Q, d)^\perp$, $f_{d, M}$ defined by $f_{d, M}(x) = \det(c_{x, M})$ using (3.2), where x are coordinates in $\text{Rep}(Q, d)$, is a relative invariant. We denote the character associated to $f_{d, M}$ by $\chi_{d, M}$. Similarly one defines relative invariants $f_{M, d}$ with associated characters $\chi_{M, d}$ for M in the left orthogonal category ${}^\perp M(Q, d)$ of $M(Q, d)$ defined by

$$\text{Hom}(M, M(Q, d)) = 0 = \text{Ext}_Q(M, M(Q, d)).$$

Assuming that Q has no oriented loops and that d is sincere, Schofield [Sch91, Thm. 4.3] proved that the latter generate the ring of all polynomial relative invariant. Due to Theorem 3.10, each irreducible component of a quiver linear free divisor D is defined by such an $f_{d, M}$ or $f_{M, d}$.

For a real Schur root $d \in \mathbb{N}^{Q_0}$, the discriminant D in $\text{Rep}(Q, d)$ is defined by

$$(3.12) \quad f_d(x) = \det(\bar{c}_x)$$

where \bar{c}_x is the map induced by c_x in (3.3) by factoring out the diagonal $\mathbb{C} \text{id}$ from $\text{End}\left(\bigoplus_{i \in Q_0} V_i\right)$. We denote by $\chi = \chi_d$ the character of $f = f_d$. By Saito’s criterion, D is a linear free divisor defined by (Q, d) exactly if f_d is reduced.

The characters of $\text{GL}(Q, d)$ are products of determinants on the factors $\text{GL}_{d_i}(\mathbb{C})$, $i \in Q_0$, and hence $X^*(\text{GL}(Q, d))$ can be identified with \mathbb{Z}^{Q_0} .

Schofield [Sch91, Lem. 1.4] and Buchweitz and Mond [BM06, Lem. 6.6] proved the following

Lemma 3.13. *Let Q be a quiver and $d \in \mathbb{N}^{Q_0}$. We have $\chi_{d,M} = Em$ for $M \in M(Q, d)^\perp$ and $\chi_{M,d} = -E^t m$ for $M \in {}^\perp M(Q, d)$, where $m = \dim(M)$. Moreover, $\chi_d = (E - E^t)d$ in case d is a real Schur root.*

We can combine this with Corollary 3.11 to obtain a formula for the degree of a polynomial relative invariant.

Proposition 3.14. *In the situation of Corollary 3.11,*

$$\deg(f_{d,M}) = \sum_{k=1}^l k \sum_{j \in Q_0^k} d_j \left(m_j - \sum_{j \rightarrow i \in Q_1 \cap (Q_0^k \times Q_0^{k+1})} m_i \right)$$

for $M \in M(Q, d)^\perp$ with $m = \dim(M)$, where we set $Q_0^{l+1} = \emptyset$. There is a similar formula for $\deg(f_{M,d})$ if $M \in {}^\perp M(Q, d)$.

Proof. The standard \mathbb{C}^* -action defines the degree of $f_{d,M}$ by $f_{d,M}(\lambda x) = \lambda^{\deg(f_{d,M})} f(x)$, and hence we have $\chi_{d,M}([\lambda]) = \lambda^{\deg(f_{d,M})}$ where $[\lambda] \in \mathrm{PGL}(Q, d) = G$ is defined by $\lambda^k \mathrm{id} \in \mathrm{GL}_{d_i}(\mathbb{C})$ for $i \in Q_0^k$, using the notation from Corollary 3.11. By Lemma 3.13 and Corollary 3.11,

$$\chi_{d,M} = Em = m - Am = \left(m_j - \sum_{j \rightarrow i \in Q_1 \cap (Q_0^k \times Q_0^{k+1})} m_i \right)_{k=1, \dots, l, j \in Q_0^k}.$$

As $\det(\lambda^k \mathrm{id}) = \lambda^{kd_i}$ for $\lambda^k \mathrm{id} \in \mathrm{GL}_{d_i}(\mathbb{C})$, each vertex $j \in Q_0^k$ contributes a summand $kd_i \left(m_j - \sum_{j \rightarrow i \in Q_0^k \times Q_0^{k+1}} m_i \right)$ to $\deg(f_{d,M})$. The claim follows. \square

3.5. Euler homogeneity. Combined with Propositions 2.9 and 3.3, Lemma 3.13 gives the following characterization of Euler homogeneity for quiver linear free divisors.

Proposition 3.15. *Let (Q, d) define a linear free divisor D and fix a Q -representation $M \in D$. Then D is Euler homogeneous at M if and only if M decomposes as $M = M_1 \oplus M_2$ such that $\langle M_1, M_2 \rangle_Q \neq \langle M_2, M_1 \rangle_Q$. In this case, M_1 and M_2 can be chosen to have no isomorphic indecomposable summands.*

Proof. By Proposition 3.3, the isotropy group of M with respect to $\mathrm{GL}(Q, d)$ is $\mathrm{Aut}_Q(M) \cong U \rtimes \prod_{i=1}^s \mathrm{GL}_{r_i}(\mathbb{C})$ where r_i are the multiplicities of the indecomposable summands of M and U is a closed normal unipotent subgroup. Then also the image of U under each homomorphism $\mathrm{GL}(Q, d) \rightarrow \mathrm{GL}_{d_i}(\mathbb{C})$, $i \in Q_0$, is unipotent and hence lies in the kernel of \det on $\mathrm{GL}_{d_i}(\mathbb{C})$. Thus U lies in the kernel of any character of $\mathrm{GL}(Q, d)$.

Whether a character is trivial on the factor $\prod_{i=1}^s \mathrm{GL}_{r_i}(\mathbb{C})$ of $\mathrm{Aut}_Q(M)$ can be checked on the torus of diagonal matrices $T = \mathbb{G}_m^s$. Indeed, $\mathrm{PGL}_{r_i}(\mathbb{C})$ is connected semisimple and has no non-trivial characters. For $i = 1, \dots, s$, let $N_i = r_i M_i$ be the product of isomorphic indecomposable summands of M , as in Theorem 3.2, and let $m_i = \dim(N_i) \in \mathbb{N}^{Q_0}$ be the corresponding dimension vector. The proof of Proposition 3.3 shows that the i th factor of T corresponds to the subgroup $\mathbb{C}^* \mathrm{id}_{N_i} \subset \mathrm{GL}(Q, d)$, $i = 1, \dots, s$. So if $\mathrm{GL}(Q, d)$ acts through a character $e \in \mathbb{Z}^{Q_0} = X^*(\mathrm{GL}(Q, d))$ then T acts through the character

$$(\langle m_i, e \rangle)_{i=1, \dots, s} \in \mathbb{Z}^s = X^*(T).$$

Now by Proposition 2.9 Euler homogeneity of D at M is equivalent to χ_d being non-trivial on $\text{Aut}_Q(M)$. This means that $0 \neq \langle m_i, e \rangle$ for some $i = 1, \dots, s$. Pick such an i for $e = \chi_d$ and set $m = m_i$ and $n = d - m_i$. Then, by Lemma 3.13, χ_d is non-trivial on $\text{Aut}_Q(M)$ if and only if

$$\begin{aligned}
 (3.13) \quad 0 &\neq \langle m, (E - E^t)d \rangle = \langle m, (E - E^t)(m + n) \rangle \\
 &= \langle m, Em \rangle - \langle m, E^t m \rangle + \langle m, En \rangle - \langle m, E^t n \rangle \\
 &= \langle m, n \rangle_Q - \langle n, m \rangle_Q
 \end{aligned}$$

for some splitting $d = m + n$ with respect to non-isomorphic indecomposable summands of M . The claim follows. \square

Remark 3.16. The “if” part of Proposition 3.15 also holds in the situation of Remark 2.10. However, it is not clear that with D also D_{red} is Euler homogeneous at M . While the module $\text{Der}(-\log D)$ is invariant under reduction, this is not clear for the property of being an Euler vector field.

Following a different approach, we can even prove the following

Theorem 3.17. *All quiver linear free divisors are strongly Euler homogeneous.*

Proof. Let (Q, d) define a linear free divisor D and fix $M = (V, f) \in D$. By Corollary 3.6.(1), M decomposes as $M = M_1 \oplus M_2 = (V_1 \oplus V_2, f_1 \oplus f_2)$. By the Ringel sequence (3.3), $\text{Ext}_Q(M) = \bigoplus_{i,j=1}^2 \text{Ext}_Q(M_i, M_j)$ can be interpreted as the normal space at M to the $\text{GL}(Q, d)$ -orbit of M in $\text{Rep}(Q, d)$, and it lifts to a subspace $W = \bigoplus_{i,j=1}^2 W_{i,j} \subset \bigoplus_{i,j=1}^2 \bigoplus_{\alpha \in Q_1} \text{Hom}_Q(V_{i,\alpha}, V_{j,\alpha})$, such that $\{M_\theta\}_{\theta \in W}$, $M_\theta := (V, f + \theta)$, is an affine space in $\text{Rep}(Q, d)$ normal to this orbit. The representation M_θ splits if $\theta_{1,2} = \theta_{2,1} = 0$. As the generic representation is indecomposable, the generic M_θ cannot split, and we must have $\text{Ext}_Q(M_1, M_2) \neq 0$ after reordering the M_1, M_2 . Consider the 1-PSG $\lambda: \mathbb{G}_m \rightarrow \text{GL}(Q, d)_M$ that acts by the characters $(1, 0) \in \mathbb{Z}^2$ on the summands. As it is non-zero on the weights of the normal representation, it induces an Euler vector field at M . \square

Combining Proposition 3.15 and Theorem 3.17 we conclude the following

Corollary 3.18. *Let (Q, d) define a linear free divisor D . Then any $M \in D$ decomposes as $M = M_1 \oplus M_2$ such that $\langle M_1, M_2 \rangle_Q \neq \langle M_2, M_1 \rangle_Q$. In this case, M_1 and M_2 can be chosen to have no isomorphic indecomposable summands.*

3.6. Local quasihomogeneity. Under additional hypotheses, the construction used in the proof of Theorem 3.17 can be used to show even weak local quasihomogeneity.

Proposition 3.19. *Let (Q, d) define a linear free divisor. Then $M = (V, f) \in D$ is weakly quasihomogeneous if $M = M_1 \oplus M_2$ with $M_1 \neq 0 \neq M_2$ and $\text{Ext}_Q(M_2, M_1) = 0$, it is quasihomogeneous if $M = M_1 \oplus \dots \oplus M_k$ with $\text{Ext}_Q(M_i, M_j) = 0$ for all $i \geq j$.*

Proof. In the first case, the 1-PSG from the proof of Theorem 3.17 is non-negative on the weights of the normal representation and the claim follows from Proposition 2.11. For the second statement, consider the 1-PSG $\lambda: \mathbb{G}_m \rightarrow \text{GL}(Q, d)_M$ that acts by the characters $(k-1, k-2, \dots, 1, 0) \in \mathbb{Z}^k$ on the summands of a decomposition $V = V_1 \oplus \dots \oplus V_k$ corresponding to $M = M_1 \oplus \dots \oplus M_k$. By hypothesis, it

is positive on the weights of the normal representation and the claim follows again from Proposition 2.11. \square

The Auslander–Reiten translation $\tau = D \operatorname{Tr}$ satisfies $\operatorname{Ext}_Q(M, N) = \operatorname{Ext}_Q(\tau M, \tau N)$ if N is a non-projective indecomposable Q -representations (see [CB92, §6, Lems. 1, 2]) and, in the case of Dynkin quivers, any indecomposable Q -representation M is preprojective, which means that $\tau^k M$ is a non-zero projective KQ -module for some k . Set $\nu(M) = k$ for the minimal such k and order the indecomposable summands M_1, \dots, M_k of M such that $\nu(M_1) \geq \dots \geq \nu(M_k)$. Consequently, $\operatorname{Ext}_Q(M_i, M_j) = \operatorname{Ext}_Q(\tau^{\nu(M_i)} M_i, \tau^{\nu(M_j)} M_j) = 0$ whenever $i \geq j$. Using Proposition 3.19, this shows the following

Theorem 3.20. *Any Dynkin quiver linear free divisor is locally quasihomogeneous.*

Remark 3.21. Fehér and Patakfalvi [FP09, §5.2] used the same argument to show more generally that all points in the representation space of a Dynkin quiver have positive normal representations.

We can generalize the preceding arguments as follows. Like τ , its inverse functor $\tau' = \operatorname{Tr} D$ leaves Ext_Q invariant and defines the class of preinjective Q -representations. Recall that a Q -representation is called *regular* if it has no preprojective or preinjective indecomposable summands. Assume that M is non-regular and, that it contains a preprojective summand M_1 or a preinjective summand M_2 . In the first case, choose M_1 such that $\nu(M_1)$ is minimal and let M' be the complement of M_1 , so that $\operatorname{Ext}_Q(M_1, M') = 0$. Similarly, $\operatorname{Ext}_Q(M', M_2) = 0$ in the second case, with M' a complement of M_2 . Then Proposition 3.19 implies the following

Theorem 3.22. *Let (Q, d) define a linear free divisor D . Then D is weakly quasihomogeneous at any non-regular point.*

3.7. The tame case. Recall that a quiver Q is called *tame* if the Tits form $q_Q(d)$, defined in (3.4), is positive and non definite. Its kernel is then the one dimensional module $\mathbb{Z} \cdot \delta$ generated by a positive sincere dimension vector $\delta \in \mathbb{N}^{Q_0}$, and characterized by the following equivalent properties:

$$\delta' \in \mathbb{Z} \cdot \delta \iff q_Q(\delta') = 0 \iff (\delta', d)_Q = \langle \delta', d \rangle_Q + \langle d, \delta' \rangle_Q = 0.$$

The goal of this section is to prove the following result.

Theorem 3.23. *Let Q be a tame quiver such that (Q, d) defines a linear free divisor D . Then D is locally weakly quasihomogeneous.*

For the rest of the section, we assume the hypotheses of Theorem 3.23. Let M be a point in D and let $M = M_1 \oplus \dots \oplus M_r$, $r \geq 2$, its decomposition into indecomposable Q -representations. According to Theorem 3.22, the only case for which local weak quasihomogeneity needs to be proved is when all the M_i are regular indecomposable. Let us recall some facts about the regular Q -representations, which can be found for example in [CB92]:

- The Auslander–Reiten functors τ and τ' applied to a regular indecomposable Q -representation X satisfy $X \cong \tau\tau'X \cong \tau'\tau X$. We define $\tau^i X$ for any $i \in \mathbb{Z}$ by

$$\tau^i X = \begin{cases} \tau^i X & \text{if } i \geq 0, \\ \tau'^{-i} X & \text{if } i \leq 0. \end{cases}$$

• Define the *defect* of an indecomposable Q -representation X to be $\text{def}(X) = \langle \delta, \dim(X) \rangle_Q = -\langle \delta, \dim(X) \rangle_Q$. Then, according to [CB92, §7, Lem. 2], the regular indecomposable Q -representations are characterized by having a zero defect:

$$X \text{ is regular} \iff \text{def}(X) = \langle \delta, \dim(X) \rangle_Q = 0$$

• The regular Q -representations form a full Abelian subcategory. We call its simple objects regular-simple.

• An indecomposable regular Q -representation X contains only a finite set of regular Q -subrepresentation X_i which are organized in a composition series

$$(3.14) \quad \{0\} = X_0 \subset X_1 \subset \cdots \subset X_r = X, \quad X_{r-i}/X_{r-i-1} = \tau^i T,$$

where T is a uniquely determined regular-simple Q -representation. We call the $\tau^i T$ the regular-simple factors of X and T the top regular-simple factor.

• Conversely for any regular-simple Q -representation $T \neq 0$ and any $r \geq 1$ there is, up to isomorphism, exactly one indecomposable regular Q -representation X with a composition series (3.14) having T as its top regular-simple factor. We denote this module by $X(T, r)$ and set $X(T, 0) = 0$. Then we have exact sequences

$$0 \longrightarrow X(\tau^s T, r-s) \longrightarrow X(T, r) \longrightarrow X(T, s) \longrightarrow 0, \quad 1 \leq s < r.$$

Lemma 3.24. *Let $X(T_1, r_1), X(T_2, r_2)$ be two regular indecomposable Q -representations.*

(1) *The space of homomorphisms $\text{Hom}_Q(X(T_1, r_1), X(T_2, r_2))$ is non zero if and only if there are integers $a, b \in \mathbb{N}$ such that $0 \leq a \leq r_1 - 1, 0 \leq b \leq r_2 - 1, a + r_2 = b + r_1$ and*

$$\tau^{r_1-1} T_1 = \tau^a (\tau^{r_2-1} T_2), \quad T_1 = \tau^b T_2.$$

(2) *The space of extensions $\text{Ext}_Q(X(T_1, r_1), X(T_2, r_2))$ is non zero if and only if there are integers $a, b \in \mathbb{N}$ such that $1 \leq a \leq r_1, 1 \leq b \leq r_2, a + r_2 = b + r_1$ and*

$$\tau^{r_2-1} T_2 = \tau^b (\tau^{r_1-1} T_1), \quad T_2 = \tau^a T_1.$$

Proof.

(1) Consider the kernel of a non zero homomorphism $\theta: X(T_1, r_1) \longrightarrow X(T_2, r_2)$. Its kernel $\ker \theta$ being again a regular Q -representation must equal $X(\tau^s T_1, r_1 - s)$, for some $s \in \mathbb{N}$ such that $1 \leq s \leq r_1$. Since $X(T_1, r_1)/X(\tau^s T_1, r_1 - s) \cong X(T_1, s)$, θ induces an injective non zero homomorphism

$$\bar{\theta}: X(T_1, s) \hookrightarrow X(T_2, r_2)$$

Its image being a regular Q -representation must equal $X(\tau^b T_2, r_2 - b)$, for some integer $b \in \mathbb{N}$ such that $0 \leq b \leq r_2 - 1$. By uniqueness of the regular-simple factors, we conclude that $\tau^{s-1} T_1 = \tau^{r_2-1} T_2$. Setting $a = r_1 - s$, this proves the last assertion.

(2) The second part follows directly from the first using the Auslander–Reiten formulas:

$$\begin{aligned} \text{Ext}_Q(X(T_1, r_1), X(T_2, r_2)) &= D(\text{Hom}_Q(X(T_2, r_2), \tau X(T_1, r_1))) \\ &= D(\text{Hom}_Q(X(T_2, r_2), X(\tau T_1, r_1))) \end{aligned}$$

□

The following periodicity result on the regular indecomposable Q -representations, which can be found in [CB92, Lec. 9], is a key ingredient of the proof of Theorem 3.23.

Proposition 3.25. *Let T be a regular-simple Q -representation. Then there is a minimal integer $p = p(T) \in \mathbb{N}$, the period of T , such that $\tau^p T \cong T$. Then*

$$(3.15) \quad \dim T + \dim \tau T + \cdots + \dim \tau^{p-1} T = \delta$$

and $X = X(T, r)$ is a brick if and only if $\dim X \leq \delta$, or equivalently $r \leq p$.

Due to this result, we may view the set of regular-simple Q -representations, as the union of disjoint cyclically ordered lists $Q_T = (T, \tau T, \dots, \tau^{p-1} T)$ of period $p = p(T)$, in which the initial T is arbitrary. Then the regular indecomposable Q -representations X are in one-to-one correspondence with the set of ordered lists $\Gamma(X) = (\tau^{a_1} T, \dots, \tau^{a_1+r-1} T)$ built from any such Q_T , the entries of $\Gamma(X)$ being the quotients X_{r-i}/X_{r-i-1} , in the sequence (3.14) where T replaced by $\tau^a T$, $0 \leq a < p$. The list $\Gamma(X)$ turns less than once, that is, $r \leq p$, if and only if X is a brick. Furthermore, $\text{Ext}_Q(X_1, X_2) \neq 0$ if and only if X_1 and X_2 are associated with the same Q_T , and we may write $\Gamma(X_i) = (\tau^{a_i} T, \dots, \tau^{a_i+r_i-1} T)$, with $a_1 < a_2 \leq a_1 + r_1 \leq a_2 + r_2 - 1$. In case $\Gamma(X_1) \cup \Gamma(X_2)$ has strictly less than p entries, this implies that this union is a list of consecutive points, and that $\Gamma(X_2)$ is obtained from $\Gamma(X_1)$ by strict turns of the both the initial and terminal point in direction of τ . With this picture in mind, we are ready for the proof of Theorem 3.23. We shall write $d < \delta$ if $d \leq \delta$ (component-wise) and $d \neq \delta$.

Proof of Theorem 3.23. Fix a regular point M in D with (regular) indecomposable summands X_1, \dots, X_r . Then, by linearity, $0 = \langle \delta, \dim M \rangle_Q = \langle \delta, d \rangle_Q$, which implies that the generic Q -representation $M(Q, d)$ is a regular indecomposable $X(T, r)$. Since $M(Q, d)$ is a brick by Remark 3.5 with $q_Q(d) = 1 \neq 0 = q_Q(\delta)$ by Theorem 3.4, Proposition 3.25 then implies that $d < \delta$ or equivalently $r < p$. This implies also that each $X_i = X(T_i, r_i)$ has the same property so that $r_i < p_i := p(T_i)$.

By Proposition 3.19, the claim follows if we establish the existence of a splitting $M = M_1 \oplus M_2$ such that $\text{Ext}_Q(M_2, M_1) = 0$. Let us consider a directed graph with vertices X_i and an arrow (X_i, X_j) if and only if $\text{Ext}(X_i, X_j) \neq 0$. Then the splitting can be obtained by grouping the X_i into two non trivial summands if and only if this graph does not contain any directed loop. The theorem is therefore a consequence of the following Lemma 3.26, and of the obvious fact that $\sum_{i=1}^k \dim X_i = \dim M < \delta$. \square

Lemma 3.26. *Let X_1, \dots, X_k be regular indecomposables with $\sum_{i=1}^k \dim X_i < \delta$ and $\text{Ext}(X_i, X_{i+1}) \neq 0$ for $i = 1, \dots, k-1$. Then we have $\text{Ext}(X_k, X_1) = 0$.*

Proof. According to Lemma 3.24.(2), the sets $\Gamma_i = \Gamma(X_i)$ of all Q -representations X_i lie in a single list Q_T . Moreover, the set

$$\Gamma(X_1) \cup \cdots \cup \Gamma(X_k)$$

is, due to the preliminary description, a list with consecutive entries $\tau^{a_1} T, \dots, \tau^{a_k+r_k-1} T$ starting with initial entry of $\Gamma(X_1)$ and ending with the terminal entry of $\Gamma(X_k)$. This list has a length $a_k + r_k - a_1 < p$ because of (3.15) and due to the dimension hypothesis. Then by applying Lemma 3.24.(2) again, we conclude the claim. \square

3.8. Reflection functors. Reflection functors introduced by Bernstein, Gel'fand, and Ponomarev, and independently as casting transforms by Sato and Kimura (cf. Section 2.3), play an important role in the study of quiver representations. We shall show explicitly that they induce functors on the class of (strongly Euler homogeneous) quiver linear free divisors.

Let Q be a quiver and denote by $e_k \in \mathbb{Z}^{Q_0}$ the unit vector corresponding to $k \in Q_0$. To a vertex $k \in Q_0$ with no loop attached one associates a reflection $r_k: \mathbb{Z}^{Q_0} \rightarrow \mathbb{Z}^{Q_0}$ by setting $r_k(m) = m - (m, e_k)_Q e_k$. These reflections generate the Weyl group $W_Q \subseteq \text{GL}(\mathbb{Z}^{Q_0})$ which leaves q_Q invariant.

Reflection functors are a realization on quiver representations of the above reflections on dimension vectors. Quivers Q with source k correspond to quivers $Q^* = r_k(Q)$ with sink k by reversing all arrows involving k . Dimension vectors $d \in \mathbb{N}^{Q_0}$ with

$$(3.16) \quad d_k \leq \sum_{k \rightarrow i \in Q_1} d_i$$

correspond to dimension vectors $d^* \in \mathbb{N}^{Q_0^*}$ with $d_k^* \leq \sum_{k \leftarrow i \in Q_1^*} d_i^*$ via $d^* = r_k(d)$, or more explicitly by $d_k^* = \sum_{k \rightarrow i \in Q_1} d_i - d_k$ and $d_i^* = d_i$ for $i \neq k$. Note that condition (1) in Theorem 3.4 is invariant under $d \leftrightarrow d^*$. The correspondence can be extended to a correspondence between the orbits in open subsets $\text{Rep}'(Q, d) \subset \text{Rep}(Q, d)$ and $\text{Rep}'(Q^*, d^*) \subset \text{Rep}(Q^*, d^*)$ as follows. For $M = (V_i, f_\alpha)$ and $M^* = (V_i^*, f_\alpha^*) = r_k(M)$, these are defined by the conditions that, for all k ,

$$\begin{aligned} (f_{k \rightarrow i})_{k \rightarrow i \in Q_1}: V_k &\rightarrow \bigoplus_{k \rightarrow i \in Q_1} V_i \text{ is injective and} \\ \sum_{k \leftarrow i \in Q_1^*} f_{k \leftarrow i}^*: V_k^* &\leftarrow \bigoplus_{k \leftarrow i \in Q_1^*} V_i^* \text{ is surjective.} \end{aligned}$$

Then the correspondence is given by

$$V_k^* = \text{coker}((f_{k \rightarrow i})_{k \rightarrow i \in Q_1}), \quad V_k = \ker \left(\sum_{k \leftarrow i \in Q_1^*} f_{k \leftarrow i}^* \right), \quad V_i = V_i^*, \quad i \neq k,$$

$f_{k \rightarrow i \in Q_1}^*$ and $f_{k \leftarrow i \in Q_1^*}^*$ being the canonical maps, and $f_\alpha^* = f_\alpha$ for $\alpha \in Q_1 \cap Q_1^*$.

We shall use the following trivial

Lemma 3.27. *Let Q be a quiver, k a source or sink of Q , and $d \in \mathbb{N}^{Q_0}$. If $M \in \text{Rep}(Q, d)$ is indecomposable then $M = e_k M$ or $M \in \text{Rep}'(Q, d)$. In particular, if $\#Q_0 > 1$ and d is a sincere root then $M \in \text{Rep}'(Q, d)$ and (3.16) holds true.*

Note that if (Q, d) defines a linear free divisor then the hypotheses for the last statement in Lemma 3.27 are fulfilled. Indeed, Q must have a source k by Theorem 3.10, d is sincere by definition and a Schur root by Theorem 3.4.(2), and $\#Q_0 = 1$ would imply $Q_1 = \emptyset$ and hence $\text{Rep}(Q, d) = 0$ by Theorem 3.10.

The following result can be found in [KR86, Prop. 4.1].

Theorem 3.28. *Let k be a source of a quiver Q . Then there is a homeomorphism $\text{Rep}'(Q, d)/\text{GL}(Q, d) \rightarrow \text{Rep}'(Q^*, d^*)/\text{GL}(Q^*, d^*)$ preserving the isomorphism class of endomorphism rings.*

Let us assume that (Q, d) defines a linear free divisor D and fix a source k of Q . The case of a sink k can be treated similarly. We want to show that also (Q^*, d^*)

defines a linear free divisor D^* . Let us denote by $Z = \text{Rep}(Q, d) \setminus \text{Rep}'(Q, d)$ and $Z^* = \text{Rep}(Q^*, d^*) \setminus \text{Rep}'(Q^*, d^*)$ the complements of the domain and range of the reflection functor. Note that Z and Z^* are defined by the maximal minors of $(x_{k \rightarrow i})_{k \rightarrow i \in Q_1}$ and $\sum_{k \leftarrow i \in Q_1^*} x_{k \leftarrow i}^*$ and invariant under $\text{GL}(Q, d)$ and $\text{GL}(Q, d^*)$ respectively. By Corollary 3.6.(1) and Lemma 3.27, we have $Z \subseteq D$.

Let us first assume that $d_k = \sum_{k \rightarrow i \in Q_1} d_i$. Then Z is defined by the determinant $\det((x_{k \rightarrow i})_{k \rightarrow i \in Q_1})$, and is hence a component of D . A general point in Z is a direct sum of two indecomposables N_1 and N_2 subject to the conditions in Proposition 3.7. By Lemma 3.27, this means that, after permuting N_1 and N_2 , $N_1 = e_k N_1$ and hence $\dim N_1 = 1$ by indecomposability of N_1 . As $\text{Hom}(N_2, N_1) = 0$, this implies that $e_k N_2 = 0$ and hence $1 = d_k = \sum_{k \rightarrow i \in Q_1} d_i$. In other words, there a unique arrow $\alpha: k \rightarrow l \in Q_1$ with $s\alpha = k$ and Z is defined by $x_\alpha = 0$. Note that Q^* is the full subquiver of Q with $Q_0^* = Q_0 \setminus \{k\}$ and $d \in \mathbb{N}^{Q_0}$ restricts to $d^* \in \mathbb{Z}^{Q_0^*}$. One can embed $\text{GL}(Q, d^*)$ as a direct factor into $\text{GL}(Q, d)$ by setting $g_k = g_l$ such that $\text{GL}(Q, d^*) = \text{GL}(d)_{x_\alpha}$. Thus D is of the form $D = D^* \times \mathbb{A}^1 \cup \mathbb{A}^{\dim D} \times \{0\}$ where D^* is the linear free divisor defined by (Q^*, d^*) .

We summarize these arguments as follows.

Proposition 3.29. *Let (Q, d) define a linear free divisor D . Then the following statements hold:*

- (1) Q has sources and sinks.
- (2) For a source (or sink) $k \in Q_0$, $d_k \leq \sum_{k \rightarrow i \in Q_1} d_i$ (or $d_k \leq \sum_{k \leftarrow i \in Q_1} d_i$).
- (3) In case of equality in (2), there is a unique arrow $\alpha: k \rightarrow l \in Q_1$ with $s\alpha = k$ (or $\alpha: k \leftarrow l \in Q_1$ with $t\alpha = k$) and $d_k = 1 = d_l$. Note that (Q^*, d^*) is then obtained from (Q, d) by $Q_0^* = Q \setminus \{k\}$ and $Q_1^* = Q_1 \setminus \{\alpha\}$ and $d^* = d|_{Q_0^*}$.
- (4) If (Q, d) and (Q^*, d^*) are related as in (3), then (Q, d) defines a linear free divisor if and only if (Q^*, d^*) does. Such linear free divisors are related by $D = D^* \times \mathbb{A}^1 \cup \mathbb{A}^{\dim D} \times \{0\}$.

Corollary 3.30. *Let (Q, d) define a linear free divisor D and assume that $d_i = d_j$ for all $i, j \in Q_0$. Then $d_i = 1$ for all $i \in Q_0$ and D is a normal crossing divisor.*

Now let us consider the case $d_k < \sum_{k \rightarrow i \in Q_1} d_i$. Then $Z \subset \text{Rep}(Q, d)$ and $Z^* \subset \text{Rep}(Q^*, d^*)$ are invariant determinantal subvarieties of codimension at least 2. By Theorem 3.28 and Remark (3.5), condition (2) in Theorem 3.4 is preserved under $(Q, d) \leftrightarrow (Q^*, d^*)$ and there is an open orbit in $\text{Rep}(Q^*, d^*)$ whose complement D^* is a hypersurface defined by a Saito determinant as in (3.12). Then also $Z^* \subset \text{Sing}(D^*)$ by reasons of invariance and dimension. From Theorem 3.28, one can then conclude that also condition (3) in Theorem 3.4 is preserved under $(Q, d) \leftrightarrow (Q^*, d^*)$. Thus, D^* is a linear free divisor by Theorem 3.4. We conclude the following

Theorem 3.31. *Reflection functors preserve the class of quiver linear free divisors.*

As a consequence of Corollary 3.11 and Theorems 3.14 and 3.31 we obtain the following

Corollary 3.32. *Each quiver defining a linear free divisor arises from an oriented bipartite graph by a sequence of reflection functors and adding arrows $k \rightarrow l$ with $d_k = 1 = d_l$. In other words, one can make $m = 1$ in Corollary 3.11 by such a*

sequence of operations. Moreover, we have $\deg(f_{d,M}) = \langle \pi^1(d), \pi^1(m) \rangle$ for $M \in M(Q, d)^\perp$ and $\deg(f_{M,d}) = \langle \pi^0(d), \pi^0(m) \rangle$ for $M \in {}^\perp M(Q, d)$, where $m = \dim(M)$ and $\pi^k: \mathbb{N}^{Q_0} \rightarrow \mathbb{N}^{Q_0^k}$ are the projections.

Schofield [Sch91, Thm. 4.3] showed that if M_1, \dots, M_m , $m = \#Q_1 - 1$, are the simple objects in ${}^\perp M(Q, d)$, then $f_{M_1,d}, \dots, f_{M_m,d}$ form a basis of the ring of all polynomial relative invariants. In particular, $f_{M_1,d} \cdots f_{M_r,d}$ is a reduced equation for D . Using Saito's criterion, D is then a linear free divisor if and only if $\sum_{i=1}^r \deg(f_{M_i,d}) = \deg(f_d) = \dim \text{Rep}(Q, d)$. Combined with Corollary 3.32, this proves the following

Theorem 3.33. *Let Q be a quiver with a real Schur root $d \in \mathbb{N}^{Q_0}$ and let m_1, \dots, m_r be the dimension vectors of the simple objects in $M(Q, d)^\perp$. Assume that Q is of the form as Corollary 3.11 with $m = 1$. Then D is a linear free divisor if and only if $\sum_{i=1}^r \langle \pi^1(d), \pi^1(m_i) \rangle = \dim \text{Rep}(Q, d)$. There is an analogous statement using ${}^\perp M(Q, d)$.*

3.9. Examples. We refer to [CB92, Lec. 4] for the list $\tilde{A}_n, \tilde{D}_m, \tilde{A}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ of tame quivers and the isotropic dimension vector δ of each one. According to the description in §3.7, for any dimension vector d such that $d < \delta$ the number of isomorphic classes of representations, hence the number of orbits in $\text{Rep}(Q, d)$ is finite. In the following statement we study the effect of reflection functors in the regular case. We consider tame quivers with a fixed non oriented underlying graph $|Q|$ and we allow non sincere roots. Regularity is indeed preserved by any reflection functor relative to $|Q|$ because for any such functor $d \rightarrow d^*$, we have $\delta^* = \delta$ and $\langle d, e \rangle_Q = \langle d^*, e^* \rangle_Q$, and by regularity $d^* \neq 0$ if $d \neq 0$. But we must allow reflections centered at points k such that $d_k = 0$, and we do not drop vertices where a zero appears after a reflection in order to preserve the property of regularity.

Proposition 3.34. (1) *Let Q be a tame quiver and let d be a dimension vector with $d < \delta$, and such that $M(Q, d)$ is indecomposable and regular. Then the discriminant in $\text{Rep}(Q, d)$ is a linear free divisor.*

(2) *Let us consider the set of regular linear free divisors associated with quivers Q' having the same underlying graph $|Q|$ as Q and dimension vector $d < \delta$. This set is finite and stable by reflection functors. Any regular linear free divisor can be transformed by a sequence reflections into a linear free divisor for a fixed orientation of $|Q|$.*

(3) *In the case of \tilde{E}_7 and \tilde{E}_8 , and with a fixed orientation of arrows we find respectively 20 and 28 distinct regular indecomposable dimension vectors or equivalently of distinct regular linear free divisor, and among them respectively 6 and 7 classes, up to reflections.*

Proof. (1) For the first statement we just use Lemma 3.9 which may be applied because the finiteness of the number of isomorphism classes of representations of dimension strictly less than δ implies that each irreducible component D_i of D contains an open orbit.

(2) The second statement is an easy combinatorial argument left to the reader.

(3) For the last statement we use the fact that the Auslander–Reiten functor τ is the same as the Coxeter functor obtained as the product of one reflection at each vertex of Q . Therefore the classification that we want here is just the classification under the action of τ . There are three indecomposable regular-simple dimension

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