Linear Free Divisors and Quivers

Michel Granger and David Mond

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1 Introduction

The notion of free divisor was introduced by Kyoji Saito in [14]. A reduced divisor $D = V(h) \subset \mathbb{C}^n$ is *free* if the sheaf $\mathrm{Der}(-\log D)$ of logarithmic vector fields is a locally free $\mathcal{O}_{\mathbb{C}^n}$-module. In [11] and [4], the smaller class of linear free divisors was introduced. A free divisor is *linear* if there is a basis for the module of global sections $\Gamma(\mathbb{C}^n, \mathrm{Der}(-\log D))$ consisting of vector fields all of whose coefficients in the basis $\partial/\partial x_1, \ldots, \partial/\partial x_n$ are linear functions. With respect to the standard weighting of $\mathrm{Der}(\mathbb{C}^n)$, such vector fields have weight zero, and we will refer to them in this way. Let Δ be the matrix of coefficients of such a global basis. By Saito's criterion ([14, 1.8]), $\det \Delta$ is a reduced equation for D. Thus, D is algebraic with homogeneous equation of degree n.

The best known example is the normal crossing divisor $\{x_1 \cdots x_n = 0\}$ in $\mathbb{C}^n$, with global basis of vector fields $x_1\partial/\partial x_1, \ldots, x_n\partial/\partial x_n$. Three further series of examples may be found (though not under this name) in [15].

In this paper we summarise some examples and results from [11] and [4], and give a clearer explanation (due to Michel Brion) than the explanation given in [4] for the appearance of linear free divisors as discriminants in quiver representation spaces. This enables us to describe some further structure, in Sections 5, 6 and 7.

We are grateful to Michel Brion for this explanation, and to Ragnar Buchweitz for having shown us how free divisors appear in quiver representation spaces.

2 Linear free divisors and representations of algebraic groups

Let $D \subset \mathbb{C}^n$ be a linear free divisor, let $\mathfrak{l}_D$ be the Lie algebra generated over $\mathbb{C}$ by the weight-zero vector fields, let G_D^0 be the identity component of the group $G_D \subset \mathrm{Gl}_n(\mathbb{C})$ of linear automorphisms preserving D, and let $\mathfrak{g}_D$ be its Lie algebra. Evidently the infinitesimal action of G_D determines a Lie algebra monomorphism $\mathfrak{g}_D \to \mathfrak{l}_D$. Since the integral flow of any member of $\mathfrak{g}_D$ determines a curve in G_D^0, this homomorphism is an isomorphism, and G_D is n-dimensional. It is easily seen to be algebraic. Since, outside D, $\mathrm{Der}(-\log D)$ coincides with the sheaf $\mathrm{Der}_{\mathbb{C}^n}$ of all derivations, at each point of $\mathbb{C}^n \smallsetminus D$ the tangent space to the G_D^0-orbit is equal to $\mathbb{C}^n$. It follows that $\mathbb{C}^n \setminus D$ is a single orbit of G_D^0.

So to every linear free divisor $D \subset \mathbb{C}^n$ there corresponds a connected n-dimensional algebraic subgroup $G_D^0 \subset \mathrm{Gl}_n(\mathbb{C})$ with an open orbit. However, the converse is not always true. Let G be such a group, and let Δ be the matrix of coefficients of the vector fields $\chi_1, \ldots, \chi_n$ on $\mathbb{C}^n$ generating the infinitesimal action of G. Then $\det \Delta$ is an equation for the complement D of the open orbit, which is therefore a divisor. If $\det \Delta$ is reduced (i.e. square-free), then by Saito's criterion D is a free divisor and $\chi_1, \ldots, \chi_n$ form a (global) basis for $\mathrm{Der}(-\log D)$. But if $\det \Delta$ is not reduced then D can never be a linear free divisor - its reduced equation has degree less than n.

Conclusion Linear free divisors in $\mathbb{C}^n$ are in bijection with connected algebraic subgroups G of $\mathrm{Gl}_n(\mathbb{C})$ satisfying

(Dim) $\dim G = n$ **(O)** G has an open orbit **(Red)** $\det \Delta$ is reduced.

Representations satisfying **(O)** are known as *prehomogenenous spaces*.

3 Examples

3.1 Irreducible linear free divisors

In [15], irreducible prehomogeneous spaces are classified, up to the action of a "castling transformation" closely related to the Bernstein-Gelfand- Ponomarev reflection functors introduced in [3] and described in Section 7 below. The class of prehomogeneous spaces satisfying **(Dim)**, **(O)** and **(Red)** is closed under the operation of castling, and thus each example gives rise to an infinite series. The simplest of these arises as the discriminant in the (four-dimensional) space of binary cubics ([15, §5 Proposition 6]). In this guise it is sometimes known as the *umbilic bracelet*. Representations with discriminant a linear free divisor giving rise to two further castling classes are described in [15, §5 Propositions 11 and 15].

When it is a divisor, the complement of the open orbit in an irreducible prehomogeneous space is necessarily irreducible ([15, §4 Proposition 12]). Besides $\{0\} \subset \mathbb{C}$, the examples of Sato and Kimura are the only irreducible linear free divisors that we know. In Section 4 below, we explain how linear free divisors arise as discriminants in quiver representation spaces. Among these only $0 \subset \mathbb{C}$ is irreducible.

By a classical theorem of E.Cartan, if the faithful representation of G on V is irreducible, then G is reductive. All the groups G_D in Sato-Kimura's list are therefore reductive. The groups we will obtain in quiver representations are also reductive. Before turning to quivers, we therefore give one series of non-reductive example. Many more are given in [11, Subsection 4.2].

3.2 The Borel subgroup of $\mathbf{Gl}_n(\mathbb{C})$

The group B_n of upper triangular matrices acts on the space Sym_n of symmetric $n \times n$ matrices by transpose conjugation, $B \cdot S = B^t SB$. This action evidently satisfies **(Dim)**; it satisfies **(O)** because, for example, the isotropy of the point I_n (the identity matrix) is $B_n \cap \mathrm{O(n)} = \{\mathrm{id}\}$. Let D be the complement of the open orbit. To see that the representation also satisfies **(Red)**, we reason as follows: for any $n \times n$ matrix S, let S_j be the $j \times j$ top left hand corner of S. Since $(B \cdot S)_j = B_j \cdot S_j$, it follows that the divisor $\{\det S_j = 0\}$ is preserved by the action of B_n. Hence $\cup_{j=1}^n \{\det S_j = 0\} \subset D$. But the degree of $\cup_{j=1}^n \{\det S_j = 0\}$ is equal to the dimension of Sym_n and thus to the degree of $\det \Delta$. It follows that $\det \Delta$ is reduced. The group B_n is of course not reductive; indeed, in the representation we describe here, $\{S_j = 0\}$ has no invariant complement.

4 Linear free divisors in quiver representation spaces

A quiver Q is an oriented graph, consisting of a collection Q_0 of nodes and a collection Q_1 of arrows. A representation V of a quiver is the assignation of a space $V(x)$ for each $x \in Q_0$ and a linear map $V(\alpha) : V(t\alpha) \to V(h\alpha)$ for each $\alpha \in Q_1$. Here $t\alpha$ (the tail of α) denotes the node at which α begins, and $h\alpha$ (the head of α) the node where α ends. If V and W are representations of Q then a *morphism* of representations $V \to W$ is a set of linear maps $\big(\varphi_x : V(x) \to W(x)\big)_{x\in Q_0}$ satisfying the commutation relations

$$\varphi_{h\alpha} \circ V(\alpha) = W(\alpha) \circ \varphi_{t\alpha}$$

for all $\alpha \in Q_1$. The representations of Q and their morphisms form an abelian category, $\mathrm{Rep}(Q)$. It is not hard to show (cf [1, Proposition 1.4 page 52]) that this category is *hereditary*: every subobject of a projective object is projective. From this it follows that every object (representation) has a projective resolution of length no more than 1, so that $\mathrm{Ext}^i_Q(V, W) = 0$ for all $V, W \in \mathrm{Rep}(Q)$ and all $i \geq 2$.

A representation of Q is *indecomposable* if it is not isomorphic to a direct sum in $\mathrm{Rep}(Q)$. The *dimension vector* of a representation V is the vector $(\dim V(x))_{x\in Q_0}$. Given a dimension vector $\mathbf{d} = (d_x)_{x\in Q_0} \in \mathbb{N}^{Q_0}$, we can form the vector space $\mathrm{Rep}(Q, \mathbf{d})$ consisting of all representations in which $V(x) = \mathbb{C}^{d_x}$. Every representation with dimension vector $\mathbf{d}$ is isomorphic to some (non-unique) member of $\mathrm{Rep}(Q, \mathbf{d})$.

The group $\mathrm{Gl}_{Q,\mathbf{d}} := \prod_{x\in Q_0} \mathrm{Gl}_{d_x}(\mathbb{C})$ acts on $\mathrm{Rep}(Q, \mathbf{d})$ by

$$(\varphi_x)_{x\in Q_0} \cdot \big(V(\alpha)\big)_{\alpha\in Q_1} = \big(\varphi_{h\alpha} \circ V(\alpha) \circ \varphi_{t\alpha}^{-1}\big)_{\alpha\in Q_1}.$$

It is (almost) this group acting on this space that will play the role of G^0_D. But first we have to factor out a 1-dimensional central subgroup Z_0 which always acts trivially: the subgroup of those $\big(\varphi_x\big)_{x\in Q_0}$ for which each φ_x is a scalar matrix $\lambda\mathrm{id}$, with the same λ for all $x \in Q_0$. We denote by $\mathbb{P}\,\mathrm{Gl}_{Q,\mathbf{d}}$ the quotient $\mathrm{Gl}_{Q,\mathbf{d}}/Z_0$. This group acts faithfully on $\mathrm{Rep}(Q, \mathbf{d})$, provided Q is connected. Now **(Dim)** holds if and only if $\dim \mathbb{P}\,\mathrm{Gl}(Q, \mathbf{d}) = \dim \mathrm{Rep}(Q, \mathbf{d})$, and thus if and only if

$$\sum_{x\in Q_0} (d_x)^2 - \sum_{\alpha\in Q_1} d_{t\alpha} d_{h\alpha} = 1. \tag{1}$$

The left hand side of (1) is a quadratic form in $\mathbf{d}$; the associated (non-symmetric) bilinear form

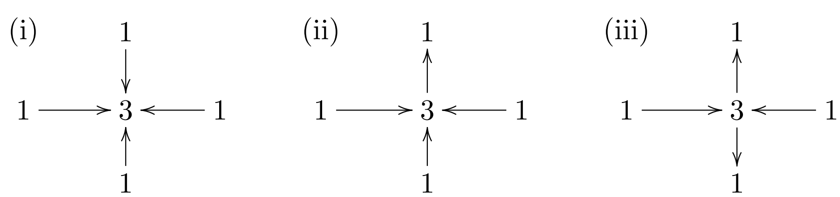
$$\langle \mathbf{d}, \mathbf{e}\rangle := \sum_{x\in Q_0} d_x e_x - \sum_{\alpha\in Q_1} d_{t\alpha} e_{h\alpha} \tag{2}$$

is known as the Euler form, and plays a central rôle in the theory, as we shall see in the next section.

Example 4.1. If Q is any quiver whose underlying graph is a tree and $\mathbf{d}$ is the dimension vector assigning the dimension 1 to each vertex, then $\langle \mathbf{d}, \mathbf{d}\rangle = 1$, so **(Dim)** holds. It is easy to see that the other conditions hold too, and that the discriminant D is a normal crossing divisor.

In general (1) is not a sufficient condition for **(O)**; we have also to ensure that there is at least one point $V \in \mathrm{Rep}(Q, \mathbf{d})$ (and hence an open set of points) whose isotropy (in $\mathbb{P}\,\mathrm{Gl}_{Q,\mathbf{d}}$) is finite. Clearly, indecomposability of V is a necessary condition for this to hold: if $V = V_1 \oplus V_2$ then each of V_1 and V_2 contributes at least a 1-dimensional subgroup to the isotropy of V in $\mathrm{Gl}_{Q,\mathbf{d}}$.

Example 4.2. ([4, Example 7.5]) Consider the three quivers and dimension vectors



In the representation space of each one, **(Dim)** holds - it is independent of the orientation of the arrows. In (i), **(O)** and **(Red)** also hold and D is a linear free divisor. In (ii), **(O)** holds but **(Red)** fails. In (iii), **(O)** fails - the generic representations decompose. Many more examples, with explicit equations, are given in [4, §7-10].

Definition 4.3. The dimension vector $\mathbf{d}$ is a *root* of the quiver Q if $\mathrm{Rep}(Q, \mathbf{d})$ contains an indecomposable representation, and a *Schur root* if the generic representation in $\mathrm{Rep}(Q, \mathbf{d})$ is indecomposable. It is a *real root* if up to isomorphism there is only one indecomposable representation in $\mathrm{Rep}(Q, \mathbf{d})$.

In Example 4.2 the dimension vector shown is a real root for all three quivers, but only a Schur root for the first and second.

If $\mathbf{d}$ is a real Schur root then the representation of $\mathbb{P}\,\mathrm{Gl}_{Q,\mathbf{d}}$ on $\mathrm{Rep}(Q,\mathbf{d})$ satisfies **(O)**. In fact, if $\mathbf{d}$ is a real Schur root then **(Dim)** is also satisfied. This is because if $V \in \mathrm{Rep}(Q,\mathbf{d})$ is stably indecomposable (meaning that all representations in a neighbourhood of V are indecomposable) then its isotropy in $\mathbb{P}\,\mathrm{Gl}_{Q,\mathbf{d}}$ is trivial (see e.g. [12]). The main theorem (Corollary 5.5) of [4] was the statement that if Q is a Dynkin quiver — meaning that the underlying unoriented graph of Q is a Dykin diagram of type A_n, D_n or E_6, E_7 or E_8 — then when $\mathbf{d}$ is a real Schur root, **(Red)** is also satisfied, so that the complement of the open orbit is a linear free divisor. In section 5.1 below we give a considerably simpler proof, pointed out to us by Michel Brion, than was given in [4].

4.1 The fundamental exact sequence

Let V and W be representations of a quiver Q. In [13], Ringel constructed an exact sequence

$$0 \to \mathrm{Hom}_Q(W,V) \to \mathrm{Hom}_{\mathbb{C}}(W,V) \xrightarrow{M_{W,V}} \mathrm{Hom}_{\mathbb{C}}(tW,hV) \xrightarrow{E_{W,V}} \mathrm{Ext}^1(W,V) \to 0. \tag{3}$$

Here

$$\mathrm{Hom}_{\mathbb{C}}(W,V) = \bigoplus_{x\in Q_0} \mathrm{Hom}_{\mathbb{C}}(W(x),V(x))$$

$$\mathrm{Hom}_{\mathbb{C}}(tW,hV) = \bigoplus_{\alpha\in Q_1} \mathrm{Hom}_{\mathbb{C}}(W(t\alpha),V(h\alpha)),$$

and the map $M_{W,V}$ sends $(\varphi_x)_{x\in Q_0}$ to $\big(\varphi_{h\alpha}\circ W(\alpha) - V(\alpha)\circ\varphi_{t\alpha}\big)_{\alpha\in Q_1}$. It is evident that $\ker M_{W,V} = \mathrm{Hom}_Q(W,V)$ since $M_{W,V}$ simply measures commutativity of the diagrams (one for each $\alpha \in Q_1$)

$$\begin{array}{ccc} W(t\alpha) & \xrightarrow{W(\alpha)} & W(h\alpha) \\ \downarrow{\scriptstyle \varphi_{t\alpha}} & & \downarrow{\scriptstyle \varphi_{h\alpha}} \\ V(t\alpha) & \xrightarrow[V(\alpha)]{} & V(h\alpha) \end{array} \tag{4}$$

The map $E_{W,V}$ sends $(\psi_\alpha)_{\alpha\in Q_1}$ to an extension

$$0 \to V \xrightarrow{i} X \xrightarrow{j} W \to 0 \tag{5}$$

where $X(x) = V(x) \oplus W(x)$ for each $x \in Q_0$, i and j are the usual inclusion and projection, and for each $\alpha \in Q_1$, $X(\alpha)$ has matrix

$$\begin{pmatrix} V(\alpha) & -\psi_\alpha \\ 0 & W(\alpha) \end{pmatrix} \tag{6}$$

Exactness of (3) can be checked by a straightforward calculation.

It is useful to consider the case where $W = V$. For then $\mathrm{Hom}_{\mathbb{C}}(V,V) = \bigoplus_{x\in Q_0} \mathrm{End}(V_x)$ is naturally identified with the Lie algebra $\mathfrak{gl}_{Q,\mathbf{d}}$, $\mathrm{Hom}_{\mathbb{C}}(tV,hV)$ is naturally identified with the space of infinitesimal deformations of V, and under these identifications $M_{V,V}$ is the infinitesimal action of $\mathfrak{gl}_{Q,\mathbf{d}}$ associated with the action of $\mathrm{Gl}_{Q,\mathbf{d}}$ on the space of representations. That is, if $\varphi_x(t)$ is a curve in $\mathrm{Gl}_{Q,\mathbf{d}}$ based at id, then

$$M_{V,V}\Big(\frac{d}{dt}\big(\varphi_x(t)\big)|_{t=0}\Big) = \frac{d}{dt}\big(\varphi(t)\cdot V\big)|_{t=0}.$$

In particular, $\mathrm{Ext}^1_Q(V,V)$ is the normal space to the $\mathrm{Gl}_{Q,\mathbf{d}}$ orbit through V, and vanishes if and only if V is rigid - i.e. has an open orbit.

The sequence is the key to Schofield's method ([16]) for finding equations for the divisor D.

5 Schofield's method

Suppose that the vector space V is prehomogeneous for the action of the connected group G, and that the complement D of the open orbit in V is a divisor. If D_i is an irreducible component of D and has reduced equation h_i then h_i is a semi-invariant for the contragredient action of G on $\mathbb{C}[V]$ — that is, there is a linear character $\chi : G \to \mathbb{C}$ such that for $v \in V$ and $g \in G$, $h_i(gv) = \chi(g)h_i(v)$. For we must have $g(D_i) = D_i$ (remember that G is connected), and so $h_i \circ g$ is a scalar multiple of h_i. The scalar is easily seen to be a linear character. Conversely, if $h \in \mathbb{C}[V]$ is a semi-invariant with associated character χ, then $\{h = 0\}$ is an invariant hypersurface, and thus contained in D. It follows that we can find equations for D by looking at the semi-invariants of the action of G on $\mathbb{C}[V]$. Sato and Kimura show in [15] that if $h_1, \ldots, h_s$ are the irreducible factors of the equation of D then the subring of $\mathbb{C}[V]$ generated by the semi-invariants is equal to $\mathbb{C}[h_1, \ldots, h_s]$.

Now let $V \in \mathrm{Rep}(Q, \mathbf{d})$ and $W \in \mathrm{Rep}(Q, \mathbf{e})$. The morphism $M_{W,V} : \mathrm{Hom}_{\mathbb{C}}(W, V) \to \mathrm{Hom}_{\mathbb{C}}(tW, hV)$ of Ringel's exact sequence is equivariant with respect to the natural action of $\mathrm{Gl}_{Q,\mathbf{e}} \times \mathrm{Gl}_{Q,\mathbf{d}}$. Crucially, if $\langle \mathbf{e}, \mathbf{d} \rangle = 0$ then $M_{W,V}$ has a square matrix. Let $P(W, V) = \det M_{W,V}$. Because of the equivariance of $M_{W,V}$, $P(W, V)$, considered as a polynomial on $\mathrm{Rep}(Q, \mathbf{e}) \times \mathrm{Rep}(Q, \mathbf{d})$, is a semi-invariant for the action of $\mathrm{Gl}_{Q,\mathbf{e}} \times \mathrm{Gl}_{Q,\mathbf{d}}$. It may, of course, be identically zero. In fact since $\ker M_{W,V} = \mathrm{Hom}_Q(W, V)$, $P(W, V) = 0$ if and only if $\mathrm{Hom}_Q(W, V) \neq 0$. We are interested in semi-invariants for the action of $\mathrm{Gl}_{Q,\mathbf{d}}$ on $\mathrm{Rep}(Q, \mathbf{d})$, and so we fix W and write $P_W(V)$ in place of $P(W, V)$. We have established

Lemma 5.1. *Let $\mathbf{d}$ be a real Schur root of Q, let $\langle \mathbf{e}, \mathbf{d} \rangle = 0$, and let $W \in Rep(Q, \mathbf{e})$. Then P_W is non-trivial if and only if for generic $V \in Rep(Q, \mathbf{d})$, $Hom_Q(W, V) = 0$.* □

By the exactness of Ringel's sequence (3), any two of the following three statements imply the third:

$$(1)\ \langle \mathbf{e}, \mathbf{d} \rangle = 0 \qquad (2)\ \mathrm{Hom}_Q(W, V) = 0 \qquad (3)\ \mathrm{Ext}^1_Q(W, V) = 0$$

It follows that if $\mathbf{d}$ is a real Schur root of Q, then to find equations for the discriminant D in $\mathrm{Rep}(Q, \mathbf{d})$, we look for the *left perpendicular category*, ${}^{\perp}V$, of the generic representation $V \in \mathrm{Rep}(Q, \mathbf{d})$ — that is, for representations W such that

$$\mathrm{Hom}_Q(W, V) = 0 = \mathrm{Ext}^1_Q(W, V).$$

Schofield shows in [16] that the irreducible components of D are defined by the polynomials P_W coming from the simple objects in ${}^{\perp}V$. In fact, as Derksen and Weyman point out in [6], if $W_1 \hookrightarrow W$ is an inclusion in ${}^{\perp}V$, then P_{W_1} divides P_W. More precisely, the cokernel W_2 of this inclusion also lies in ${}^{\perp}V$, and from the short exact sequence $0 \to W_1 \to W \to W_2$ it follows that $P_W = P_{W_1} P_{W_2}$. It was shown by both Schofield [16] and Geigle and Lenzing ([9]) that ${}^{\perp}V$ is equivalent to the category of all representations of some quiver $Q({}^{\perp}V)$ with one fewer vertices than Q; in $\mathrm{Rep}(Q({}^{\perp}V))$ there are $|Q_0| - 1$ simple objects (the 1-dimensional representations concentrated at a single node). Thus D has $|Q_0| - 1$ irreducible components, a fact first proved by V. Kac. A topological proof is given below, in Section 8.

It is sometimes transparently clear that the existence of a homomorphism $W \to V$ implies a codimension 1 degeneracy in V. In the quiver (i) of Example 4.2 with the dimension vector shown, the discriminant has four components. On each, three of the four arrows have coplanar images. Coplanarity of the images of A, B, C is evidently equivalent to the injectivity of the homomorphism φ shown here, where W is a generic (rigid) representation for the dimension-vector shown.

$$\begin{array}{ccccccccccc} W & & 0 & & & & V & & 1 & & \\ & & \downarrow & & & & & & \downarrow & & \\ 1 & \longrightarrow & 2 & \longleftarrow & 1 & \xrightarrow{\ \varphi\ } & 1 & \xrightarrow{A} & 3 & \xleftarrow{B} & 1 \\ & & \uparrow & & & & & & C\uparrow & & \\ & & 1 & & & & & & 1 & & \end{array} \tag{7}$$

With larger dimension vectors or more complicated quivers this transparency may be lost. Nevertheless the fact (see 7.6 below) that the following two (generic) representations are perpendicular, gives us a clear geometrical description of a degeneracy for the representations on the right: there should exist a 3-dimensional subspace of the space at the central node, containing the image of one of the four maps (here D), and meeting each of the other three images in a line.

$$\begin{array}{ccccc} W & & 2 & & \\ & & \downarrow & & \\ 1 & \longrightarrow & 3 & \longleftarrow & 1 \\ & & \uparrow & & \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} V & & 2 & & \\ & & \downarrow D & & \\ 2 & \xrightarrow{A} & 5 & \xleftarrow[B]{} & 2 \\ & & C\uparrow & & \\ & & 2 & & \end{array} \tag{8}$$

5.1 Vanishing extensions and the question of reducedness

Let $\mathbf{d}$ be a real Schur root of the quiver Q with divisor D complementary to the open orbit in $\mathrm{Rep}(Q,\mathbf{d})$. We have already seen that **(Dim)** and **(O)** hold, so to conclude that D is a linear free divisor, we need only show that **(Red)** also holds. In [4] there is a rather long argument showing that this is the case if

(OD): each irreducible component of D is the closure of a single orbit.

Michel Brion pointed out a much simpler argument to the authors of [4],[11]. Write $\dim \mathrm{Rep}(Q,\mathbf{d}) =: N$. The hypothesis **(OD)** implies that if V_0 is a representation whose orbit is open in D, then there are weight-zero logarithmic vector fields $\delta_1,\ldots,\delta_{N-1}$ whose values at V_0 span $T_{V_0}D$. It remains to find one further logarithmic vector field δ_N such that at V_0, the determinant of the matrix of coefficients $[\delta_1,\ldots,\delta_N]$ is a reduced equation for D. The key point is that the isotropy group $G_{V_0} \subset \mathbb{P}\,\mathrm{Gl}_{Q,\mathbf{d}}$ is reductive - it is connected and 1-dimensionsional, and thus isomorphic to $\mathbb{C}^*$. It follows that the space $T_{V_0}D$, evidently invariant under G_{V_0}, has an invariant complement. The action of G_{V_0} on this complementary line ℓ is not trivial, because otherwise G_{V_0} would be contained in the isotropy of all V on the line, whereas for generic $V \in \mathrm{Rep}(Q,\mathbf{d})$, G_V is just $\{\mathrm{id}\}$. From this non-triviality it follows that the infinitesimal action of G_{V_0} gives us a vector field δ_N on $\mathrm{Rep}(Q,\mathbf{d})$, tangent to ℓ, and whose restriction to ℓ has the form $\lambda\partial/\partial t$, with respect to a suitable coordinate λ on ℓ. As ℓ is complementary to $T_{V_0}D$, this coordinate can be extended to a neighbourhood of V_0 in such a way that $\lambda = 0$ is locally a reduced equation for D. With respect to such a coordinate system, the matrix $[\delta_1,\ldots,\delta_N]$ now takes the form

$$\begin{bmatrix} * & \cdots & * & * \\ * & \cdots & * & * \\ 0 & \cdots & 0 & \lambda \end{bmatrix} \tag{9}$$

Since, at V_0, the first $N-1$ columns span the $N-1$ dimensional space $T_{V_0}D$, it follows that $\det[\delta_1,\ldots.\delta_N]$ is a reduced equation for D.

Brion's invariant line turns out to have an interesting additional significance. We now give an explicit construction of such a line, exhibiting a non-split extension whose central term is the generic representation V, which splits when V moves into the discriminant D.

Once again, let V_0 be a smooth point on D, and assume that the orbit of V_0 is open in D.

Lemma 5.2. *(i) The representation V_0 is a direct sum (in Rep(Q)) of uniquely determined submodules V_1 and V_2.*
(ii) Both submodules are rigid, and their dimension vectors $\mathbf{d_1}$ and $\mathbf{d_2}$ are real Schur roots.
(iii) After perhaps permuting V_1 and V_2, $Ext_Q(V_1,V_2) = 0$ and $Ext_Q(V_2,V_1)$ is 1-dimensional.

Proof As remarked in Section 5, $\dim \mathrm{Ext}(V_0,V_0) = \mathrm{Rep}(Q,\mathbf{d})/T_{V_0}G\cdot V_0$. Thus $\dim \mathrm{Ext}(V_0,V_0) = 1$. By the exactness of (3), $\dim \mathrm{Hom}_Q(V_0,V_0) = q(\mathbf{d}) + \dim \mathrm{Ext}^1(V_0,V_0) = 2$. The representation V_0

must split because by the definition of real Schur root $\mathbf{d}$, there is only one orbit of indecomposable representations in $\mathrm{Rep}(Q, \mathbf{d})$. As $\dim \mathrm{Hom}(V_i, V_i) \geq 1$ for any representation V_i, it follows that V_0 splits exactly into two indecomposable representations, V_1 and V_2, with $\dim \mathrm{Hom}_Q(V_1, V_1) = \dim \mathrm{Hom}_Q(V_2, V_2) = 1$ and

$$\mathrm{Hom}_Q(V_1, V_2) = \mathrm{Hom}_Q(V_2, V_1) = 0. \tag{10}$$

The representations V_1 and V_2 are unique because for $i = 1, 2$ $\mathrm{Hom}_Q(V_i, V_0)$ is 1-dimensional, and therefore must be generated by the inclusion $V_i \to V_1 \oplus V_2$. The representation V_1 is rigid because for any representation $V_1' \in \mathrm{Rep}(Q, \mathbf{d_1})$ in a sufficiently small neighborhood of V_1, the representation $V_1' \bigoplus V_2$ is in D (being decomposable) and therefore in the orbit of V_0, since this orbit is open in D. Hence by unicity, V_1 and V_1' are isomorphic. The same argument applies to V_2.

Since

$$\mathrm{Ext}^1_Q(V, V) = \bigoplus_{i,j=1,2} \mathrm{Ext}^1_Q(V_i, V_j)$$

there is only one non-zero summand on the right, and it is 1-dimensional.

By the rigidity of V_1 and V_2, $\mathrm{Ext}^1_Q(V_i, V_i) = 0$ for $i = 1, 2$. As $\dim \mathrm{Hom}_Q(V_i, V_i) = 1$ for $i = 1, 2$, it follows that $\mathbf{d}_1$ and $\mathbf{d}_2$ are real Schur roots.

We label the two representations so that

$$\dim \mathrm{Ext}^1_Q(V_2, V_1) = 1, \quad \dim \mathrm{Ext}^1_Q(V_1, V_2) = 0, \tag{11}$$

or, equivalently $\langle V_1, V_2 \rangle = 0, \quad \langle V_2, V_1 \rangle = -1 \qquad \text{and} \qquad \langle V_1, V_1 \rangle = \langle V_2, V_2 \rangle = 1.$ □

The representation V_0 is, of course, a split extension of V_2 by V_1. Where can we find a non-split extension? The answer is, arbitrarily close to V_0, as the following construction shows.

Choose $\theta \in \mathrm{Hom}_{\mathbb{C}}(tV_2, hV_1)$ so that its image under the inclusion $\mathrm{Hom}_{\mathbb{C}}(tV_2, hV_1) \to \mathrm{Hom}_{\mathbb{C}}(tV_0, hV_0)$, generates $\mathrm{Coker} M_{V_0,V_0}$. Construct a new representation $V(\lambda\theta)$ of Q, depending on the complex parameter λ, using Ringel's construction:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_1(t\alpha) & \longrightarrow & V_1(t\alpha) \oplus V_2(t\alpha) & \longrightarrow & V_2(t\alpha) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle V_1(\alpha)} & & \downarrow{\scriptstyle \begin{bmatrix} V_1(\alpha) & \lambda\theta \\ 0 & V_2(\alpha) \end{bmatrix}} & & \downarrow{\scriptstyle V_2(\alpha)} & & \\ 0 & \longrightarrow & V_1(h\alpha) & \longrightarrow & V_1(h\alpha) \oplus V_2(h\alpha) & \longrightarrow & V_2(h\alpha) & \longrightarrow & 0 \end{array} \tag{12}$$

The representation in the centre is $V(\lambda\theta)$. Evidently $V(\lambda\theta) = V_0$ when $\lambda = 0$. Since the tangent space to the line $\ell := \{V(\lambda\theta) : \lambda \in \mathbb{C}\}$ is spanned by θ, which does not belong to $\mathrm{Image}(M_{V_0,V_0}) = T_{V_0}D$, the line we get is a complement to $T_{V_0}D$. It follows that except for a finite number of values of λ, $V(\lambda\theta) \notin D$ and the extension (12) is not split, and therefore generates $\mathrm{Ext}^1_Q(V_2, V_1)$.

Lemma 5.3. *The line $\ell := \{V(\lambda\theta) : \lambda \in \mathbb{C}\}$ is invariant under the action of the isotropy group $G_{V_0} \subset Gl_{Q,\mathbf{d}}$ of V_0.*

Proof Clearly $G_{V_0} \subset \mathrm{Hom}_Q(V_0, V_0)$ so by what we have observed above, $G_{V_0} \subset \mathrm{Hom}_Q(V_1, V_1) \oplus \mathrm{Hom}_Q(V_2, V_2)$. Thus G_{V_0} is isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$ acting by scalar multiplication on each of the two summands. Let $(u, v) \in G_{V_0}$. The diagram

$$\begin{array}{ccc} V_1(t\alpha) \oplus V_2(t\alpha) & \xrightarrow{\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}} & V_1(t\alpha) \oplus V_2(t\alpha) \\ {\scriptstyle \begin{bmatrix} V_1(\alpha) & \lambda\theta \\ 0 & V_2(\alpha) \end{bmatrix}} \downarrow & & \downarrow {\scriptstyle \begin{bmatrix} V_1(\alpha) & u^{-1}v\lambda\theta \\ 0 & V_2(\alpha) \end{bmatrix}} \\ V_1(h\alpha) \oplus V_2(h\alpha) & \xrightarrow{\begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix}} & V_1(h\alpha) \oplus V_2(h\alpha) \end{array} \tag{13}$$

commutes; it follows that $(u, v) \cdot V(\lambda\theta) = V(u^{-1}v\lambda\theta)$. Thus G_{V_0} acts on ℓ. □

Theorem 5.4. *Let* $\mathbf{d}$ *be a real Schur root of the quiver* Q*, and let* $D \subset Rep(Q, \mathbf{d})$ *be the complement of the open orbit. If there is a* $Gl_{Q,d}$*-orbit which is open in the irreducible component* D_i *of* D*, then the equation* h *of the discriminant* D *is reduced along* D_i*.*

Proof We continue to use the notation and assumptions of the lemma. Regard λ as a coordinate on ℓ. Identify G_{V_0} with $\mathbb{C}^* \times \mathbb{C}^*$ as described above, and consider the curve $\sigma(u) = (1, u) \in G_{V_0}$. As $(1, u) \cdot V(\lambda\theta) = V(u\lambda\theta)$, we have

$$\frac{d}{du}\Big(\sigma(u) \cdot V(\lambda\theta)\Big)|_{u=1} = V(\lambda\theta). \tag{14}$$

Now consider $\sigma'(1)$ as an element of $\mathfrak{gl}_{Q,d}$ via the inclusion $G_{V_0} \subset \mathrm{Gl}_{Q,\mathbf{d}}$. Then (14) means that the linear vector field δ on $\mathrm{Rep}(Q, \mathbf{d})$ coming, via the infinitesimal action of $\mathrm{Gl}(Q, \mathbf{d})$, from the tangent vector $\sigma'(1)$, restricts to $\lambda\partial/\partial\lambda$ on ℓ.

Since the corank at V_0 of the map $M_{V,V}$ is one, its image generates the tangent space $T_V D$. Let us choose linear vector fields $\delta_1, \cdots, \delta_{N-1}$, where $N = \dim \mathrm{Rep}(Q, \mathbf{d})$, whose values at V_0 generates this space. The determinant $\det(\delta, \delta_1, \cdots, \delta_{N-1})$ is then reduced at V_0 since it is a non-zero multiple of λ when restricted to ℓ. Since we know also that it is a multiple of the equation of the discriminant we conclude that the germ at V_0 of this equation is reduced. □

Proposition 5.5. *Let* $\mathbf{d}$ *be a real Schur root of the quiver* Q*, and* $V_0 = V_1 \bigoplus V_2$ *the unique decomposition of the representation at a smooth point of* D*, with summands ordered as in Lemma 5.2. Then*
(i) the generic representation $V \in Rep(D, \mathbf{d})$ *is given by an extension of* V_2 *by a unique subrepresentation isomorphic to* V_1*:*

$$0 \longrightarrow V_1 \longrightarrow V \longrightarrow V_2 \longrightarrow 0 \tag{15}$$

(ii) we have $Ext_Q(V, V_i) = 0$, $Ext_Q(V_i, V) = 0$ *for* $i = 1, 2$ *and* $Hom_Q(V, V_1) = Hom_Q(V_2, V) = 0$ *and in particular* $V_2 \in {}^{\perp}V$ *and* $V_1 \in V^{\perp}$

Proof The only part of (i) which has not yet been proved is the uniqueness. This follows from the fact that $\dim \mathrm{Hom}_Q(V, V_2) = \dim \mathrm{Hom}_Q(V_1, V) = 1$, which in turn can be deduced from the equalities (10) and (1) and from the long exact sequences obtained from (15) by applying the functors $\mathrm{Hom}(V_i, \bullet)$ and $\mathrm{Hom}(\bullet, V_i)$. The equalities of (ii) are proved similarly. □

Now we change our notation slightly. To take account of the different irreducible components D_i of D, add a subindex i to the objects in the forgoing discussion. So a generic representation lying in D_i becomes V_{i0} and its canonical splitting becomes $V_{i0} = V_{i1} \oplus V_{i2}$. The simple object of ${}^{\perp}V$ whose associated semi-invariant polynomial defines D_i is now W_i.

Lemma 5.6. *(i)* $\dim Hom_Q(W_i, V_{i0}) = \dim Ext^1(W_i, V_{i0}) = 1$.
(ii) $Hom_Q(W_i, V_{i1}) = 0$
(iii) $Ext^1_Q(W_i, V_{i2}) = 0$.

Proof Recall that $P_{W_i}(V) = \det M_{W_i,V}$. At the regular point V_{i0} of $\{P_{W_i} = 0\}$, $M_{W_i,V_{i0}}$ must have corank 1. This proves the first assertion. The second follows from the fact that V_{i1} is a sub-object of the generic representation V, whereas $\mathrm{Hom}_Q(W_i, V) = 0$. Thus $\mathrm{Hom}_Q(W_i, V_{i2})$ is 1-dimensional. Now apply $\mathrm{Hom}_Q(W_i, \bullet)$ to the extension (15); because $W_i \in^{\perp} V$, this gives an isomorphism $\mathrm{Hom}_Q(W_i, V_{i2}) \simeq \mathrm{Ext}^1_Q(W_i, V_{i1})$. □

The non-trivial homomorphism $W_i \to V_{i2}$ is an injection, since its kernel also lies in ${}^{\perp}V$. Define C_i by the sequence

$$0 \to W_i \to V_{i2} \to C_i \to 0 \tag{16}$$

Since V_{i2} and C_i lie in ${}^{\perp}V$, they give rise to semi-invariant polynomials $P_{V_{i2}}$ and P_{C_i}.

Proposition 5.7. *P_{C_i} is not identically zero; $P_{V_{i2}} = P_{W_i} P_{C_i}$.*

Proof As $\mathrm{Hom}_Q(V_2, V) = 0$, $P_{V_2}(V) \neq 0$. The short exact sequence (16) in ${}^\perp V$ gives rise to the factorisation of $P_{V_{i2}}$. □

Let $\mathbf{d}$ be a real Schur root of Q and let V be a generic representation in $\mathrm{Rep}(Q, \mathbf{d})$. Construct a new quiver $Q' = Q'({}^\perp V)$ as follows:

- Q'_0 is the set of irreducible components of the discriminant D
- $a_{({}^\perp Q')}(D_j, D_i)$, the number of arrows in ${}^\perp Q'$ from D_j to D_i, is the highest power of P_{W_j} dividing $P_{V_{2,i}}$.

This quiver contains all the arrows of Schofield's quiver ${}^\perp Q$. By definition, $a_{({}^\perp Q')}(D_j, D_i)$ is the number of times W_j appears in a composition series of C_i in ${}^\perp V$. On the other hand, we have

Lemma 5.8. *$a_{({}^\perp Q)}(D_j, D_i) = \dim \mathrm{Hom}_Q(W_j, C_i)$.*

Proof Let E_i be the simple object in $\mathrm{Rep}({}^\perp Q)$ concentrated at vertex D_i. For $i \neq j$, $a_{{}^\perp Q}(D_j, D_i) = -\langle E_j, E_i \rangle_{{}^\perp Q}$. By [16, Theorem 2.4], $\langle W_j, W_i \rangle_Q = \langle E_j, E_i \rangle_{{}^\perp Q}$. As $i \neq j$, $\mathrm{Hom}_Q(W_j, W_i) = 0$. Hence

$$a_{({}^\perp Q)}(D_j, D_i) = \dim \mathrm{Ext}_Q(W_j, W_i).$$

Since $j \neq i$, $\mathrm{Hom}_Q(W_j, V_{i0}) = \mathrm{Ext}_Q(W_j, V_{i0}) = 0$, and thus also $\mathrm{Hom}_Q(W_j, V_{i2}) = \mathrm{Ext}_Q(W_j, V_{i2}) = 0$. By applying $\mathrm{Hom}_Q(W_j, \_)$ to the exact sequence (16), we deduce that

$$\mathrm{Hom}_Q(W_j, C_i) \simeq \mathrm{Ext}_Q(W_j, W_i).$$ □

5.2 Dynkin quivers

A quiver Q is a *Dynkin quiver* if its underlying graph is a Dynkin diagram of type A_n, D_n, E_6, E_7 or E_8. Gabriel proved in [8] that the Dynkin quivers are precisely those of "finite representation type": that is, such that for any $\mathbf{d}$, $\mathrm{Rep}(Q, \mathbf{d})$ contains only finitely many orbits. For such quivers **(OD)** always holds, and thus by Brion's argument, described in the preceding section, we conclude

Theorem 5.9. *([4]) If $\mathbf{d}$ is a real Schur root of the Dynkin quiver Q then the discriminant D in Rep$(Q, \mathbf{d})$ is a linear free divisor.* □

6 Where are the vanishing cycles?

In singularity theory one expects the rank of the homology of geometrical objects to diminish when they degenerate (acquire singularities). With surprising frequency, the rank of the vanishing homology (the Milnor number, in the most classical case) is closely related (even equal) to the dimension of the miniversal base space of the degeneration, the Tjurina number. The discriminant in the versal base space is the set of parameter values for which the rank of the homology of the corresponding fibre is less than maximal. This leads us to the question heading this section. What homology theory would allow us to associate to the degenerations of representations a 'vanishing homology'? In Section 5.1 we associated, to each irreducible component of the discriminant, something that behaves like a vanishing cycle: the extension (15), which splits when V moves onto the discriminant. This suggests that we should associate to each $V \in \mathrm{Rep}(Q, \mathbf{d})$ some 'homology' derived from the (poset of) sub-representations.

7 BGP reflection functors

In [3], Bernstein, Gelfand and Ponomarev re-proved and explained Gabriel's theorem characterising the finite type quivers as Dynkin quivers. Their principal tools were a pair of "reflection functors" which we now describe.

Let Q be a quiver and let $x \in Q_0$ be a sink - a node at which some arrows arrive and none depart. Let V be a representation of Q. Let Q' be a new quiver differing from Q only in that all the arrows arriving at x are reversed. Denote the arrows of Q by α, β, etc., and the arrows of Q' by $\alpha', \beta', \ldots$. We make use of the evident bijection between the two sets of arrows. A representation V' of Q' is constructed as follows:

$$\begin{array}{ll} V'(y) = V(y) & \text{if } y \in Q_0 \setminus \{x\} \\ V'(x) = \ker\Big(\bigoplus_{\alpha \in h^{-1}(x)} V(t\alpha) \xrightarrow{\sum V(\alpha)} V(x)\Big) & \\ V'(\alpha') = V(\alpha) & \text{if } t\alpha' \neq x \\ V'(\beta') = \text{ composite of } \quad V'(x) \to \bigoplus_{\alpha \in h^{-1}(x)} V(t\alpha) \xrightarrow{\text{pr}} V'(h\beta') & \text{if } t\beta' = x \end{array} \tag{17}$$

We denote the representation V' obtained in this way by $R_x^h(V)$ (the 'R' stands for 'reflection' and the 'h' for 'head').

If x is a source - a node from which some arrows depart but none arrive - then a new quiver Q' is defined by reversing all of the arrows leaving x, and a new representation V' is defined by

$$\begin{array}{ll} V'(y) = V(y) & \text{if } y \in Q_0 \setminus \{x\} \\ V'(x) = \text{Coker}\Big(V(x) \to \bigoplus_{\alpha \in t^{-1}(x)} V(h\alpha)\Big) & \\ V'(\alpha') = V(\alpha) & \text{if } t\alpha \neq x \\ V'(\beta') = \text{ composite of } \quad V'(t\beta') \to \bigoplus_{\alpha \in t^{-1}(x)} V(h\alpha) \to V'(x) & \text{if } h\beta' = x \end{array} \tag{18}$$

We denote the representation V' obtained in this way by $R_x^t(V)$.

We call R_x^h and R_x^t "BGP reflections". Where it is not necessary to distinguish between the two, we simply speak of R_x. A node x which is either a sink or a source in Q will be referred to as *monotone*. In [3] it is shown that if V is an indecomposable representation of Q and x is a monotone node then $R_x(V)$ is also indecomposable, unless V is supported only at x, in which case $R_x(V) = 0$. Both R_x^h and R_x^t are functors: it is easily seen that a morphism $\varphi \in \text{Hom}_Q(V_1, V_2)$ gives rise to $R_x(\varphi) \in \text{Hom}_{Q'}(R_x(V_1), R_x(V_2))$, and $R_x(\varphi_1 \circ \varphi_2) = R_x(\varphi_1) \circ R_x(\varphi_2)$.

The reflections R_x^h and R_x^t can be applied to dimension vectors: if x is a sink, then $R_x^h(\mathbf{d})$ is the dimension vector $\mathbf{d}'$ defined by

$$d'_y = d_y \quad \text{if} \quad y \neq x \qquad\qquad d'_x = \sum_{\alpha \in h^{-1}(x)} d_{t\alpha} - d_x \tag{19}$$

It is the dimension vector of $R_x^h(V)$ if V is indecomposable in $\text{Rep}(Q, \mathbf{d})$, unless V is supported only at x. For, with this exception, the indecomposability of V implies that $\Big(\bigoplus_{\alpha \in h^{-1}(x)} V(t\alpha) \xrightarrow{\sum V(\alpha)} V(x)\Big)$ is an epimorphism. Similarly, if x is a source and $\mathbf{d}$ is a dimension vector then $R_x^t(\mathbf{d})$ is the dimension vector $\mathbf{d}'$ defined by

$$d'_y = d_y \quad \text{if} \quad y \neq x \qquad\qquad d'_x = \sum_{\alpha \in t^{-1}(x)} d_{h\alpha} - d_x \tag{20}$$

and is the dimension vector of $R_x^t(V)$ unless V is supported only at x. The Euler form (2) is invariant by this transformation :

Lemma 7.1. ([3, Lemma 2.1]) *Let* $\mathbf{d}$ *and* $\mathbf{e}$ *be dimension vectors and let* $\mathbf{d}' = R_x(\mathbf{d})$ *and* $\mathbf{e}' = R_x(\mathbf{e}')$ *be their images by a BGP reflection, Then* $\langle \mathbf{e}, \mathbf{d} \rangle = \langle \mathbf{e}', \mathbf{d}' \rangle$ □

Let Q be a quiver without oriented cycles, and choose an ordering $x_1, \ldots, x_n$ of the vertices compatible with the arrows, i.e. so that for each arrow α, $t\alpha \geq h\alpha$. The composite functors $C^t := R^t_{x_1} \circ \cdots \circ R^t_{x_n}$ and $C^h := R^h_{x_n} \circ \cdots \circ R^h_{x_1}$ are called *Coxeter functors* in [3], and are independent of the choice of ordering compatible with the arrows. They map $\mathrm{Rep}(Q)$ to itself.

The crucial point in the proof of Gabriel's theorem in [3] is the fact that it is only for Dynkin quivers that the group generated by the Coxeter transformations acting on the space of dimension vectors is finite. As we have seen, this leads both to the conclusion that if D is a Dynkin quiver then for every real Schur root $\mathbf{d}$ the discriminant in $\mathrm{Rep}(Q, \mathbf{d})$ is a linear free divisor, and that only finitely many linear free divisors arise in this way. Nevertheless, functoriality means that if $\mathbf{d}$ is a real Schur root for a non-Dynkin quiver for which the discriminant in $\mathrm{Rep}(Q, \mathbf{d})$ is a linear free divisor, then by applying the BGP reflection functors we get infinitely many more. We now explain this in more detail.

Proposition 7.2. *Let* $\mathbf{d}$ *be a real Schur root with indecomposable representation* $V(\mathbf{d})$*. Let* R_x *be a BGP reflection such that* $R_x(\mathbf{d}) \neq \mathbf{0}$*. Then* $R_x(\mathbf{d})$ *is also a real Schur root, and* $V(R_x(\mathbf{d})) = R_x(V(\mathbf{d}))$*.*

Proof By [3, Theorem 1.1], $R_x(V(\mathbf{d}))$ is indecomposable with dimension vector $R_x(\mathbf{d})$ and, moreover, applying the BGP transformation R'_x with same center we get the equality $R'_x(R_x(V(\mathbf{d}))) = V(\mathbf{d})$. From this it follows that there is an isomorphism

$$\mathrm{Hom}_{Q'}(V(\mathbf{d}), V(\mathbf{d})) \to \mathrm{Hom}_Q(R_x(V(\mathbf{d})), R_x(V(\mathbf{d}))),$$

where $Q' = R_x(Q)$. Both spaces are then one dimensional and since $q(\mathbf{d}) = q(R_x(\mathbf{d})) = 1$ this implies $\mathrm{Ext}_{Q'}(R_x(V(\mathbf{d})), R_x(V(\mathbf{d}))) = 0$ so that $R_x(V(\mathbf{d}))$ is rigid as required. □

From now on we assume that $\mathbf{d}$ is a sincere real Schur root, ($d_x \neq 0$ for all $x \in Q_0$) and that $R_x(\mathbf{d})$ is also sincere.

The simple objects in ${}^\perp V(\mathbf{d})$ are the representations $W_i := V(\mathbf{e_i})_{1 \leq \ell n-1}$, for some real Schur roots $\mathbf{e_i}$. Because $\mathbf{d}$ is sincere, $R_x(\mathbf{e_i}) \neq \mathbf{0}$ so that by Proposition 7.2, $R_x(\mathbf{e_i})$ is a real Schur root and $R_x(V(\mathbf{e}_i)) = V(R_x(\mathbf{e}_i)) \in {}^\perp R_x(V(\mathbf{d}))$. Similarly, because $R_x(\mathbf{d})$ is sincere, $R'_x(R_x(W_i)) \neq 0$ and therefore by [3, 1.1] $R'_x R_x(W_i) = W_i$.

Let $V := V(\mathbf{d})$ and $W_i := V(\mathbf{e_i})$. According to Schofield, $f_i = det(M_{W_i,V})$ are the irreducible reduced equations of the components of D.

Proposition 7.3. *If* D *satisfies* **(OD)***, then so does the divisor* D' *obtained by the BGP reflection* R_x*.*

Lemma 7.4. *The invariants* $P_{W'_i} = det(M_{W'_i,V'})$*, are the irreducible equations of the components of* D'

Proof We have to prove that the W'_i are the simple objects of ${}^\perp V'$. We deal with the case where $R_x = R^h_x$. The case of R^t_x is similar.

We know that $W_i = R^t_x(R^h_x)(W_i) = R^t_x(W'_i)$. If W'_i was not simple there would exist a non trivial exact sequence

$$0 \longrightarrow W' \longrightarrow W'_i \longrightarrow W'' \longrightarrow 0$$

in ${}^\perp V'$. Being a quotient is preserved by the functor R^t_x. (Dually, being a submodule is preserved by R^h_x). Therefore $R^t_x(W'')$ is a quotient of $R^t_x(W'_i) = W_i$. Since W_i is simple, $R^t_x(W'') = 0$ or $R^t_x(W'') = W_i$, and the representation W'' in the first case, or W' in the second case, must be concentrated at x. These conclusions contradict the relations $\langle W'', V' \rangle = 0$ and $\langle W', V' \rangle = 0$, since $d'_x - \sum_{h\alpha'=x} d'_{t\alpha} = \dim V_x \neq 0$. □

Proof of 7.3: We continue with the case of R_x^h. The argument for R_x^t is similar. We have to show that if $V'_{i,0} = V'_{i,1} \bigoplus V'_{i,2}$ is a general point in the component D'_i of of the divisor D' then $\mathrm{Ext}Q'(V'_{i,0}, V'_{i,0})$ is one-dimensional. We will prove it by showing that $V'_{i0} \simeq R_x(V_{i0})$, in two steps.

(i) First, $\dim R_x(V_{i0}) = R_x(\mathbf{d})$, so that $R_x(V_{i0})$ is an element of $\mathrm{Rep}(Q, R_x(\mathbf{d}))$.

(ii) Second, $\dim \mathrm{Hom}_{Q'}(R_x(V_{i0}), R_x(V_{i0})) = 2$.

By [3, Lemma 1.1], $R_x(V_{i,0}) = R_x(V_{i,1}) \oplus R_x(V_{i,2})$. By [3, Theorem 1.1], $R_x(V_{i,1})$ and $R_x(V_{i,2})$ are indecomposable with the expected dimension vectors $R_x(\mathbf{d}_1)$, $R_x(\mathbf{d}_2)$, from which (i) follows, unless V_{i1} or V_{i2} is the simple representation E_x concentrated at x. In the case of a sink ($R_x = R_x^h$), E_x cannot be a quotient of V because the composition of $\bigoplus_{h\alpha=x} V_{t\alpha} \to V_x \to E_x$ would have to be be zero and onto at the same time. This implies that $E_x = V_{i,1}$. But by 5.5, $V_{i,1} \in V^\perp$, and this would give $\langle V, E_x\rangle = d_x - \sum_{h\alpha=x} d_{t\alpha} = 0$ contradictioning the fact that $R_x(\mathbf{d})$ is sincere. So (i) holds. Now by (i) we know that $R_x^t \circ R_x^h(V_{i0}) = V_{i0}$, and the map

$$\mathrm{Hom}_Q(V_{i0}, V_{i0}) \to \mathrm{Hom}_{Q'}(R_x^h(V_{i0}), R_x^h(V_{i0}))$$

is bijective, with inverse given by the functor R_x^t. This proves (ii). It follows that both Hom spaces are two-dimensional, and thus that $\dim \mathrm{Ext}(R_x^h(V_i0), R_x^h(V_i0)) = 1$. Hence **(OD)** holds for the component of D' containing V'_{i0}. It remains only to show that this component is D'_i.

This holds because for each W_j, $\dim(\mathrm{Hom}_{Q'}(R_x^h(W_j), R_x^h(V_{i0}))) = \dim(\mathrm{Hom}_Q(W_j, V_{i0}))$ is distinct from zero only when $j = i$. This proves that $R_x^h(V_{i0})$ is in D'_i and no other D'_j. □

Example 7.5. Consider the star quiver Q of Example 4.2, with all arrows pointing in to the central node. Order Q_0 so that the central node is the zero'th, and label each arrow by the node at its tail. The dimension vector $\mathbf{d} = (3, 1, 1, 1, 1)$ gives rise to a linear free divisor in a space of dimension 12. The vector $\mathbf{e} = (2, 1, 1, 1, 0)$ is a real Schur root whose generic representation $V(\mathbf{e})$ is in ${}^\perp V(\mathbf{d})$, and gives rise to the semi-invariant $\det[\alpha_1, \alpha_2, \alpha_3]$. This representation and the three others obtained by permuting the last four entries in $\mathbf{e}$ are the simple objects of ${}^\perp V(\mathbf{d})$.

Applying repeatedly to these representations the Coxeter functor C^t. From 7.3 we deduce :

Proposition 7.6. *For $n \geq 1$ let*

$$\mathbf{d_n} = (2n+1, n, n, n, n) \qquad \mathbf{e_n} = \big(n+1, [\frac{n+1}{2}], [\frac{n+1}{2}], [\frac{n+1}{2}], [\frac{n+1}{2}] + (-1)^n\big).$$

Then
(i) The discriminant in Rep($Q, \mathbf{d}_n$) is a linear free divisor of degree $8n^2 + 4n$.
(ii) $V(\mathbf{e}_n) \in {}^\perp V(d_n)$, and $P_{V(\mathbf{e}_n)}$ is one of the four equations defining the irreducible components of the discriminant in Rep($Q, \mathbf{d}_n$). The other three are obtained similarly, after permuting the last four entries in $\mathbf{e}_n$. □

8 Logarithmic differential forms

We summarise a discussion in [11]. If D is a linear free divisor with group G_D^0, then the weight-zero part of $\Gamma(\mathbb{C}^n, \mathrm{Der}(-\log D))$ is isomorphic to $\mathfrak{g}_D$. It follows that the weight-zero part of $\Gamma(\mathbb{C}^n, \Omega^1(\log D))$ is isomorphic to $\mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}_D, \mathbb{C})$, and this isomorphism extends to an isomorphism

$$\Gamma(\mathbb{C}^n, \Omega^\bullet(\log D))_0 \simeq \bigwedge^\bullet \mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}_D, \mathbb{C}) \tag{21}$$

where the left-hand side is the weight-zero subcomplex of the complex of global sections. Now

$$H^*\big(\Gamma(\mathbb{C}^n, \Omega^\bullet(\log D))_0\big) \simeq \Gamma(\mathbb{C}^n, \Omega^\bullet(\log D)) \tag{22}$$

by an argument which uses the Lie derivative with respect to the Euler vector field as a contracting homotopy. Moreover if G_D^0 is reductive,

$$H^*\big(\bigwedge^{\bullet}\mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}_D,\mathbb{C})\big)\simeq H^*(G_D^0;\mathbb{C}). \tag{23}$$

Since $\mathbb{C}^n \smallsetminus D$ is a finite quotient of the connected group G_D^0,

$$H^*(G_D^0)\simeq H^*(\mathbb{C}^n\smallsetminus D;\mathbb{C}). \tag{24}$$

Putting (22), (23) and (24) together we conclude that the complex of global differential forms with logarithmic poles along D calculates the cohomology of $\mathbb{C}^n \smallsetminus D$; this is summarised by saying that the *global logarithmic comparison theorem* (or GLCT for short) holds for D. A stronger (local as well as global) version is known to hold if D is a locally weighted homogeneous free divisor ([5]). GLCT was conjectured by Terao in [17] for all hyperplane arrangements.

Let $\mathbf{d}$ be a real Schur root of a quiver Q, and let D be the discriminant in $\mathrm{Rep}(Q,\mathbf{d})$. Then $\mathbb{P}\,\mathrm{Gl}_{Q,\mathbf{d}}$ is reductive (it is a central quotient of a product of general linear groups). If **(Red)** holds then we deduce that GLCT holds for D. Even if **(Red)** fails, (23) and (24) still hold, and so

$$H^*\big(\bigwedge^{\bullet}\mathrm{Hom}_{\mathbb{C}}(\mathfrak{g}_D,\mathbb{C})\big)\simeq H^*(\mathrm{Rep}(Q,\mathbf{d})\smallsetminus D;\mathbb{C}). \tag{25}$$

In particular the number of irreducible components of D, being equal to dim $H^1(\mathrm{Rep}(Q,\mathbf{d})\smallsetminus D;\mathbb{C})$, is dim $H^1\big(\bigwedge^{\bullet}\mathrm{Hom}_{\mathbb{C}}(\mathfrak{g},\mathbb{C})\big)$. Since $H^1(\mathfrak{gl}_n(\mathbb{C}))=\mathbb{C}$, $H^1(\mathrm{Gl}_{Q,\mathbf{d}})\simeq\mathbb{C}^{|Q_0|}$. Taking the quotient by Z_0, we reduce the dimension of H^1 by 1; it follows that the number of irreducible components of D is equal to $|Q_0|-1$, as mentioned earlier in our discussion of Schofield's method.

9 Linear Free Divisors and Mirror Symmetry

Givental ([10]) and Barannikov ([2]), later complemented by Sabbah and Douai ([7]), described a Frobenius structure on the base-space of the miniversal deformation of the function $f(x_1,\ldots,x_{n+1})=x_1+\cdots+x_{n+1}$ on the Milnor fibre, $X_t, t\neq 0$, of the normal crossing divisor $X_0=\{x_1\cdots x_{n+1}=0\}$, and showed that this structure is isomorphic to the natural Frobenius structure on the small quantum cohomology of $\mathbb{C}\,\mathbb{P}^n$. Let $g(x)=x_1\cdots x_{n+1}$ be the equation of X_0 and let $\mathrm{Der}(-\log g)$ be the $\mathcal{O}_{\mathbb{C}^{n+1}}$-module of vector fields annihilating g. The Jacobian algebra of $f: X_t\to\mathbb{C}$ is the quotient

$$A_t(g):=J_{f:X_1\,\to\,\mathbb{C}}=\mathbb{C}[x_1,\ldots,x_{n+1}]/\big(df(\mathrm{Der}(-\log g)+(g-t)\big), \tag{26}$$

and is spanned by the 0'th, ..., n'th powers of f. It is this algebra structure, and its deformations, which are transferred to the tangent bundle of the base space by the Kodaira-Spencer map of the deformation. When $t=0$, the algebra defined by (26) becomes the *classical* cohomology of $\mathbb{C}\,\mathbb{P}^n$.

Let $D_0=\{h=0\}$ be a linear free divisor in $\mathbb{C}^{n+1}$. Under a simple non-degeneracy hypothesis, which holds, for example, if G_D^0 is reductive, for almost all linear functions $f(x)=\sum_j a_jx_j$ the family of algebras $A_t(h)$ defined by replacing g in (26) by h, is isomorphic to the family $\{A_t(g)\}$. Ignacio de Gegorio has shown that these give rise to the same F-manifold structures (as in [10], [2], [7]) on the miniversal base-space of $f: D_t\to\mathbb{C}$ for $t\neq 0$, and observes that, once again, by including the case $t=0$, we obtain a degeneration of this structure to the trivial (constant) Frobenius structure on the classical cohomology of $\mathbb{C}\,\mathbb{P}^n$. We do not know whether the Frobenius structure which presumably can be defined on the base space of $f: D_t\to\mathbb{C}$ for $t\neq 0$ is also isomorphic to that of [10], [2], [7].

Finally, let us recall that if Q is any quiver whose underlying graph is a tree then the dimension vector $\mathbf{d}(1)$ taking the value 1 at each node is the smallest sincere real Schur root, and the resulting discriminant in $\mathrm{Rep}(Q,\mathbf{d}(1))$ is a normal crossing divisor. We do not know whether, among the Frobenius manifolds constructed by the procedure of the previous paragraph from the discriminants associated with the other real Schur roots, we will find a reflection of the intricate combinatorial universe of quiver representations.

References

[1] M.Auslander, I.Reiten, S. Smalø, *Representation Theory of Artin Algebras*, Studies in Advanced Math. 36, Cambridge University Press, 1995

[2] S. Barannikov, *Semi-infinite Hodge structures and mirror symmetry for projective spaces*, AG/0108148, 2000

[3] I.N. Bernstein, I.M. Gel'fand, V.A. Ponomarev, *Coxeter functors and Gabriel's theorem*, Uspekhi Mat. Nauk. 28 (1973), 19-33, translated in Russian Math. Surveys 28 (1973) 17-32

[4] R.-O.Buchweitz and D. Mond, *Free divisors and quiver representations*, in *Singularities and Computer Algebra*, C.Lossen and G.Pfister Eds, London Math. Soc. Lecture Notes in Math. 324, Cambridge University Press, 2006, pp. 41-77, available as math.AG/05090221

[5] F. Castro, D. Mond, L. Narváez, *Cohomology of the complement of a free divisor*, Trans. Amer. Math. Soc. 348 (1996) 3037-3049

[6] H.Derksen and J.Weyman, *Semi-invariants of quivers and saturation for Littlewood Richardson coefficients*, J.Amer. Math.Soc. 13 (2000) 467-479

[7] A. Douai and C.Sabbah, *Gauss Manin systems, Brieskorn lattices and Frobenius structures I*, in *Proc. Int. Conf. in Honour of F.Pham, Nice 2002*, Ann. Inst. Fourier (Grenoble) 53, No. 4, 2003, 1055-1116

[8] P. Gabriel, *Unzerlegbare Darstellungen I*, Manuscripta Math. 6 (1972) 71-103

[9] W. Geigle and H. Lenzing, *Perpendicular categories with applications to representations and sheaves*, J. Algebra 144 (1991) no. 2, 273-343

[10] A. Givental, *A mirror theorem for toric complete intersections*, in *Topological field theory, primitive forms and related topics, Kyoto 1996*, Prog. Math. 160, Birkhäuser Boston, 1998, 141-175

[11] Michel Granger, David Mond, Alicia Nieto and Mathias Schulze, *Linear Free Divisors*, submitted, available at `www.maths.warwick.ac.uk/ mond/`

[12] H. Kraft and Ch. Riedtmann, *Geometry of representations of quivers*, in *Representations of Algebras, Durham 1985*, P.Webb (Ed.), London Math. Soc. Lecture Notes 116, 1986, 109-146

[13] C.M.Ringel, *Representations of K species and bimodules*, J.Algebra 41 (1976) 269-302

[14] K. Saito, *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. Univ. Tokyo Sect. Math. 27 (1980), 265-291

[15] M. Sato and T. Kimura, *A classification of irreducible prehomogeneous vector spaces and their relative invariants*, Nagoya J. Math. 65 (1977) 1-155

[16] A.Schofield, *Semi-invariants of quivers*, J.London Math. Soc. (2) 43 (1991) 385-395

[17] H. Terao, *Forms with logarithmic poles and the filtration by the order of the pole*, Proc. Int. Sympos. Alg. Geom., Kyoto, 1977, Kinokuniya, Tokyo, 1978, 673-685