STRING TOPOLOGY FOR SPHERES.

LUC MENICHI*
WITH AN APPENDIX BY GERALD GAUDENS AND LUC MENICHI

Abstract. Let M be a compact oriented d-dimensional smooth manifold. Chas and Sullivan have defined a structure of Batalin-Vilkovisky algebra on $\mathbb{H}_*(LM)$. Extending work of Cohen, Jones and Yan, we compute this Batalin-Vilkovisky algebra structure when M is a sphere S^d, $d \geq 1$. In particular, we show that $\mathbb{H}_*(LS^2;\mathbb{F}_2)$ and the Hochschild cohomology $HH^*(H^*(S^2);H^*(S^2))$ are surprisingly not isomorphic as Batalin-Vilkovisky algebras, although we prove that, as expected, the underlying Gerstenhaber algebras are isomorphic. The proof requires the knowledge of the Batalin-Vilkovisky algebra $H_*(\Omega^2S^3;\mathbb{F}_2)$ that we compute in the Appendix.

Dedicated to Jean-Claude Thomas, on the occasion of his 60th birthday

1. Introduction

Let M be a compact oriented d-dimensional smooth manifold. Denote by $LM := map(S^1, M)$ the free loop space on M. In 1999, Chas and Sullivan [2] have shown that the shifted free loop homology $\mathbb{H}_*(LM) := H_{*+d}(LM)$ has a structure of Batalin-Vilkovisky algebra (Definition 5). In particular, they showed that $\mathbb{H}_*(LM)$ is a Gerstenhaber algebra (Definition 8). This Batalin-Vilkovisky algebra has been computed when M is a complex Stiefel manifold [25] and very recently over $\mathbb{Q}$ when M is a $K(\pi,1)$ [28]. In this paper, we compute the Batalin-Vilkovisky algebra $\mathbb{H}_*(LM;\Bbbk)$ when M is a sphere S^n, $n \geq 1$ over any commutative ring $\Bbbk$ (Theorems 10, 16, 17, 24 and 25).

In fact, few calculations of this Batalin-Vilkovisky algebra structure or even of the underlying Gerstenhaber algebra structure have been done because the following conjecture has not yet been proved.

Conjecture 1. *(due to* [2, “dictionary” p. 5] *or* [7]*?)*

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If M is simply connected then there is an isomorphism of Gerstenhaber algebras $\mathbb{H}_*(LM) \cong HH^*(S^*(M); S^*(M))$ between the free loop space homology and the Hochschild cohomology of the algebra of singular cochains on M.

In [7, 5], Cohen and Jones proved that there is an isomorphism of graded algebras over any field

$$\mathbb{H}_*(LM) \cong HH^*(S^*(M); S^*(M)).$$

Over the reals or over the rationals, two proofs of this isomorphism of graded algebras have been given by Merkulov [23] and Félix, Thomas, Vigué-Poirrier [11]. Motivated by this conjecture, Westerland [30] has computed the Gerstenhaber algebra $HH^*(S^*(M; \mathbb{F}_2); S^*(M; \mathbb{F}_2))$ when M is a sphere or a projective space.

What about the Batalin-Vilkovisky algebra structure?

Suppose that M is formal over a field, then since the Gerstenhaber algebra structure on Hochschild cohomology is preserves by quasi-isomorphism of algebras [10, Theorem 3], we obtain an isomorphism of Gerstenhaber algebras

$$HH^*(S^*(M); S^*(M)) \cong HH^*(H^*(M); H^*(M)). \tag{2}$$

Poincaré duality induces an isomorphism of $H^*(M)$-modules

$$\Theta : H^*(M) \to H^*(M)^\vee.$$

Therefore, we obtain the isomorphism

$$HH^*(H^*(M); H^*(M)) \cong HH^*(H^*(M); H^*(M)^\vee)$$

and the Gerstenhaber algebra structure on $HH^*(H^*(M); H^*(M))$ extends to a Batalin-Vilkovisky algebra [26, 22, 20] (See above Proposition 20 for details). This Batalin-Vilkovisky algebra structure is further extended in [27, 9, 19, 21] to a richer algebraic structure. It is natural to conjecture that this Batalin-Vilkovisky algebra on $HH^*(H^*(M); H^*(M))$ is isomorphic to the Batalin-Vilkovisky algebra $\mathbb{H}_*(LM)$. We show (Corollary 30) that this is not the case over $\mathbb{F}_2$ when M is the sphere S^2. See [6, Comments 2. Chap. 1] or the papers of Tradler and Zeinalian [26, 27] for a related conjecture when M is not assumed to be necessarily formal. On the contrary, we prove (Corollary 23) that Conjecture 1 is satisfied for $M = S^2$ over $\mathbb{F}_2$.

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2. The Batalin-Vilkovisky algebra structure on $\mathbb{H}_*(LM)$.

In this section, we recall the definition of the Batalin-Vilkovisky algebra on $\mathbb{H}_*(LM;\Bbbk)$ given by Chas and Sullivan [2] over any commutative ring $\Bbbk$ and deduce that this Batalin-Vilkovisky algebra $\mathbb{H}_*(LM;\Bbbk)$ behaves well with respect to change of rings.

We first recall the definition of the loop product following Cohen and Jones [7, 6]. Let M be a closed oriented smooth manifold of dimension d. The inclusion $e : map(S^1 \vee S^1, M) \hookrightarrow LM \times LM$ can be viewed as a codimension d embedding between infinite dimension manifolds [24, Proposition 5.3]. Denote by ν its normal bundle. Let $\tau_e : LM \times LM \twoheadrightarrow map(S^1 \vee S^1, M)^\nu$ its Thom-Pontryagin collapse map. Recall that the umkehr (Gysin) map $e_!$ is the composite of τ_e and the Thom isomorphism:

$$H_*(LM\times LM;\Bbbk) \overset{H_*(\tau_e;\Bbbk)}{\to} H_*(map(S^1\vee S^1, M)^\nu;\Bbbk) \underset{\cong}{\overset{\cap u_\Bbbk}{\to}} H_{*-d}(map(S^1\vee S^1, M);\Bbbk)$$

The Thom isomorphism is given by taking a relative cap product $\cap$ with a Thom class for ν, $u_\Bbbk \in H^d(map(S^1 \vee S^1, M)^\nu;\Bbbk)$. A Thom class with coefficients in $\mathbb{Z}$, $u_\mathbb{Z}$, gives rise to a Thom class $u_\Bbbk$ with coefficients in $\Bbbk$, under the morphism

$$H^d(map(S^1 \vee S^1, M);\mathbb{Z}) \to H^d(map(S^1 \vee S^1, M);\Bbbk)$$

induced by the ring homomorphism $\mathbb{Z} \to \Bbbk$ [16, p. 441-2]. So we have the commutative diagram

$$\begin{array}{ccc} H_*(LM \times LM;\mathbb{Z}) & \overset{e_!}{\longrightarrow} & H_{*-d}(map(S^1 \vee S^1, M);\mathbb{Z}) \\ \downarrow & & \downarrow \\ H_*(LM \times LM;\Bbbk) & \overset{e_!}{\longrightarrow} & H_{*-d}(map(S^1 \vee S^1, M);\Bbbk) \end{array}$$

Let $\gamma : map(S^1 \vee S^1, M) \to LM$ be the map obtained by composing loops. The loop product is the composite

$$H_*(LM;\Bbbk) \otimes H_*(LM;\Bbbk) \to H_*(LM \times LM;\Bbbk)$$
$$\overset{e_!}{\to} H_{*-d}(map(S^1 \vee S^1, M);\Bbbk) \overset{H_{*-d}(\gamma;\Bbbk)}{\to} H_{*-d}(LM;\Bbbk)$$

So clearly, we have proved

Lemma 3. *The morphism of abelian groups $\mathbb{H}_*(LM;\mathbb{Z}) \to \mathbb{H}_*(LM;\Bbbk)$ induced by $\mathbb{Z} \to \Bbbk$ is a morphism of graded rings.*

Suppose that the circle S^1 acts on a topological space X. Then we have an action of the algebra $H_*(S^1)$ on $H_*(X)$,

$$H_*(S^1) \otimes H_*(X) \to H_*(X).$$

Denote by $[S^1]$ the fundamental class of the circle. Then we define an operator of degree 1, $\Delta : H_*(X;\mathbb{k}) \to H_{*+1}(X;\mathbb{k})$ which sends x to the image of $[S^1] \otimes x$ under the action. Since $[S^1]^2 = 0$, $\Delta \circ \Delta = 0$. The following lemma is obvious.

Lemma 4. *Let X be a S^1-space. We have the commutative diagram*

$$\begin{array}{ccc} H_*(X;\mathbb{Z}) & \xrightarrow{\Delta} & H_{*+1}(X;\mathbb{Z}) \\ \downarrow & & \downarrow \\ H_*(X;\mathbb{k}) & \xrightarrow{\Delta} & H_{*+1}(X;\mathbb{k}) \end{array}$$

where the vertical maps are induced by the ring homomorphism $\mathbb{Z} \to \mathbb{k}$.

The circle S^1 acts on the free loop space on M by rotating the loops. Therefore we have a operator Δ on $\mathbb{H}_*(LM)$. Chas and Sullivan [2] have showed that $\mathbb{H}_*(LM)$ equipped with the loop product and the Δ operator, is a Batalin-Vilkovisky algebra.

Definition 5. A *Batalin-Vilkovisky algebra* is a commutative graded algebra A equipped with an operator $\Delta : A \to A$ of degree 1 such that $\Delta \circ \Delta = 0$ and

$$(6) \quad \Delta(abc) = \Delta(ab)c + (-1)^{|a|}a\Delta(bc) + (-1)^{(|a|-1)|b|}b\Delta(ac) - (\Delta a)bc - (-1)^{|a|}a(\Delta b)c - (-1)^{|a|+|b|}ab(\Delta c).$$

Consider the bracket $\{\ ,\ \}$ of degree $+1$ defined by

$$\{a, b\} = (-1)^{|a|} \left(\Delta(ab) - (\Delta a)b - (-1)^{|a|}a(\Delta b)\right)$$

for any a, $b \in A$. (6) is equivalent to the following relation called the *Poisson relation*:

$$(7) \qquad \{a, bc\} = \{a, b\}c + (-1)^{(|a|+1)|b|}b\{a, c\}.$$

Getzler [14, Proposition 1.2] has shown that the $\{\ ,\ \}$ is a Lie bracket and therefore that a Batalin-Vilkovisky algebra is a Gerstenhaber algebra.

Definition 8. A *Gerstenhaber algebra* is a commutative graded algebra A equipped with a linear map $\{-,-\} : A \otimes A \to A$ of degree 1 such that:
a) the bracket $\{-,-\}$ gives to A a structure of graded Lie algebra of degree 1. This means that for each a, b and $c \in A$

$\{a,b\} = -(-1)^{(|a|+1)(|b|+1)}\{b,a\}$ and
$\{a,\{b,c\}\} = \{\{a,b\},c\} + (-1)^{(|a|+1)(|b|+1)}\{b,\{a,c\}\}$.
b) the product and the Lie bracket satisfy the Poisson relation (7).

Using Lemma 3 and Lemma 4, we deduce

Proposition 9. *The $\Bbbk$-linear map*

$$\mathbb{H}_*(LM;\mathbb{Z}) \otimes_{\mathbb{Z}} \Bbbk \hookrightarrow \mathbb{H}_*(LM;\Bbbk)$$

is an inclusion of Batalin-Vilkovisky algebras.

In particular, by the universal coefficient theorem,

$$\mathbb{H}_*(LM;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{H}_*(LM;\mathbb{Q}).$$

More generally, this Proposition tell us that if $\mathrm{Tor}^{\mathbb{Z}}(\mathbb{H}_*(LM;\mathbb{Z}),\Bbbk) = 0$ then the Batalin-Vilkovisky algebra $\mathbb{H}_*(LM;\mathbb{Z})$ determines the Batalin-Vilkovisky algebra $\mathbb{H}_*(LM;\Bbbk)$.

3. The circle and an useful Lemma.

In this section, we compute the structure of the Batalin-Vilkovisky algebra on the homology of the free loop space on the circle S^1 using a Lemma which gives information on the image of Δ on elements of lower degree in $H_*(LM)$.

Theorem 10. *As a Batalin-Vilkovisky algebra, the homology of the free loop space on the circle is given by*

$$\mathbb{H}_*(LS^1;\Bbbk) \cong \Bbbk[\mathbb{Z}] \otimes \Lambda a_{-1}.$$

Denote by x a generator of $\mathbb{Z}$. The operator Δ is

$$\Delta(x^i \otimes a_{-1}) = i(x^i \otimes 1), \quad \Delta(x^i \otimes 1) = 0$$

for all $i \in \mathbb{Z}$.

Let X be a pointed topological space. Consider the free loop fibration $\Omega X \overset{j}{\hookrightarrow} LX \overset{ev}{\twoheadrightarrow} X$. Denote by $hur_X : \pi_n(X) \to H_n(X)$ the Hurewicz map.

Lemma 11. *Let $n \in \mathbb{N}$. Let $f \in \pi_{n+1}(X)$. Denote by $\tilde{f} \in \pi_n(\Omega X)$ the adjoint of f. Then* [1]

$$(H_*(ev) \circ \Delta \circ H_*(j) \circ hur_{\Omega X})(\tilde{f}) = hur_X(f).$$

[1] Added in 2014: Since the homology suspension σ_* is the composite $H_*(ev) \circ \Delta \circ H_*(j)$, this Lemma is well-known: McCleary Lemma 6.11.

Proof. Take in homology the image of $[S^1]\otimes[S^n]$ in the following commutative diagram

$$\begin{array}{ccccc} S^1\times\Omega X & \xrightarrow{S^1\times j} & S^1\times LX & \xrightarrow{act_{LX}} & LX \\ {\scriptstyle S^1\times\tilde{f}}\uparrow & & & & \downarrow{\scriptstyle ev} \\ S^1\times S^n & \longrightarrow & S^1\wedge S^n & \xrightarrow[f]{} & X \end{array}$$

where $act_{LX}: S^1\times LX\to LX$ is the action of the circle on LX. □

Proof of Theorem 10. More generally, let G be a compact Lie group. Consider the homeomorphism $\Theta_G:\Omega G\times G\overset{\cong}{\to} LG$ which sends the couple (w,g) to the free loop $t\mapsto w(t)g$. In fact, Θ_G is an isomorphism of fiberwise monoids. Therefore by [15, part 2) of Theorem 8.2],

$$\mathbb{H}_*(\Theta_G): H_*(\Omega G)\otimes\mathbb{H}_*(G)\to\mathbb{H}_*(LG)$$

is a morphism of graded algebras. Since $H_*(S^1)$ has no torsion,

$$\mathbb{H}_*(\Theta_{S^1}): H_*(\Omega S^1)\otimes\mathbb{H}_*(S^1)\cong\mathbb{H}_*(LS^1)$$

is an isomorphism of algebras. Since Δ preserves path-connected components,

$$\Delta(x^i\otimes a_{-1})=\alpha(x^i\otimes 1)$$

where $\alpha\in\Bbbk$. Denote by $\varepsilon_{\Bbbk[\mathbb{Z}]}$ the canonical augmentation of the group ring $\Bbbk[\mathbb{Z}]$. Since $H_*(ev\circ\Theta_{S^1})=\varepsilon_{\Bbbk[\mathbb{Z}]}\otimes H_*(S^1)$,

$$(H_*(ev)\circ\Delta)(x^i\otimes a_{-1})=\alpha 1.$$

On the other hand, applying Lemma 11, to the degree i map $S^1\to S^1$, we obtain that $(H_*(ev)\circ\Delta\circ H_*(j))(x^i)=i1$. Therefore $\alpha=i$. □

4. Computations using Hochschild homology.

In this section, we compute the Batalin-Vilkovisky algebra $\mathbb{H}_*(LS^n)$, $n\geq 2$, using the following elementary technique:

The algebra structure has been computed by Cohen, Jones and Yan using the Serre spectral sequence [8]. On the other hand, the action of $H_*(S^1)$ on $H_*(LS^n)$ can be computed using Hochschild homology. Using the compatibility between the product and Δ, we determine the Batalin-Vilkovisky algebra $\mathbb{H}_*(LS^n)$ up to isomorphism. This elementary technique will fail for $\mathbb{H}_*(LS^2)$.

Let A be an augmented differential graded algebra. Denote by $s\overline{A}$ the suspension of the augmentation ideal $\overline{A}$, $(s\overline{A})_i=\overline{A}_{i-1}$. Let d_1 be the differential on the tensor product of complexes $A\otimes T(s\overline{A})$. The (normalized) Hochschild chain complex, denoted $\mathcal{C}_*(A;A)$, is the complex $(A\otimes T(s\overline{A}), d_1+d_2)$ where

$$\begin{aligned} d_2 a[sa_1|\cdots|sa_k] =&(-1)^{|a|}aa_1[sa_2|\cdots|sa_k]\\ &+\sum_{i=1}^{k-1}(-1)^{\varepsilon_i}a[sa_1|\cdots|sa_ia_{i+1}|\cdots|sa_k]\\ &-(-1)^{|sa_k|\varepsilon_{k-1}}a_ka[sa_1|\cdots|sa_{k-1}]; \end{aligned}$$

Here $\varepsilon_i=|a|+|sa_1|+\cdots+|sa_i|$.

Connes boundary map B is the map of degree $+1$

$$B: A\otimes (s\overline{A})^{\otimes p}\to A\otimes (s\overline{A})^{\otimes p+1}$$

defined by

$$B(a_o[sa_1|\dots|sa_p])=\sum_{i=0}^{p}(-1)^{|sa_0\dots sa_{i-1}||sa_i\dots sa_p|}[sa_i|\dots|sa_p|sa_0|\dots|sa_{i-1}].$$

Up to the isomorphism $s^p(A^{\otimes(p+1)})\to A\otimes(sA)^{\otimes p}$, $s^p(a_0[a_1|\dots|a_p])\mapsto(-1)^{p|a_0|+(p-1)|a_1|+\cdots+|a_{p-1}|}a_0[sa_1|\dots|sa_p]$, our signs coincides with those of [29].

The Hochschild homology of A (with coefficient in A) is the homology of the Hochschild chain complex:

$$HH_*(A;A):=H_*(\mathcal{C}_*(A;A)).$$

The Hochschild cohomology of A (with coefficient in $A^\vee$) is the homology of the dual of the Hochschild chain complex:

$$HH^*(A;A^\vee):=H_*(\mathcal{C}_*(A;A)^\vee).$$

Consider the dual of Connes boundary map, $B^\vee(\varphi)=(-1)^{|\varphi|}\varphi\circ B$. On $HH^*(A;A^\vee)$, $B^\vee$ defines an action of $H_*(S^1)$.

Example 12. Let $n\geq 2$. Let $\Bbbk$ be any commutative ring. Let $A:=H^*(S^n)=\Lambda x_{-n}$ be the exterior algebra on a generator of lower degree $-n$. Denote by $[sx]^k:=1[sx|\dots|sx]$ and $x[sx]^k:=x[sx|\dots|sx]$ the elements of $\mathcal{C}_*(A;A)$ where the term sx appears k times. These elements form a basis of $\mathcal{C}_*(A;A)$. Denote by $[sx]^{k\vee}$, $x[sx]^{k\vee}$, $k\geq 0$, the dual basis. The differential $d^\vee$ on $\mathcal{C}_*(A;A)^\vee$ is given by $d^\vee([sx]^{k\vee})=0$ and $d^\vee(x[sx]^{k\vee})=\pm\left(1-(-1)^{k(n+1)}\right)[sx]^{(k+1)\vee}$. The dual of Connes boundary map $B^\vee$ is given by

$$B^\vee([sx]^{k\vee})=\begin{cases}(-1)^{n+1}k\ x[sx]^{(k-1)\vee} & \text{if } (k+1)(n+1) \text{ is even,}\\ 0 & \text{if } (k+1)(n+1) \text{ is odd}\end{cases}$$

and $B^\vee(x[sx]^{k\vee})=0$. We remark that $[sx]^{k\vee}$ is of (lower) degree $k(n-1)$ and $x[sx]^{k\vee}$ of degree $n+k(n-1)$.

Theorem 13. [17] *Let X be a simply connected space such that $H_*(X;\mathbb{k})$ is of finite type in each degree. Then there is a natural isomorphism of $H_*(S^1)$-modules between the homology of the free loop space on X and the Hochschild cohomology of the algebra of singular cochain $S^*(X;\mathbb{k})$:*

$$H_*(LX)\cong HH^*(S^*(X;\mathbb{k});S^*(X;\mathbb{k})^\vee). \tag{14}$$

In this paper, when we will apply this theorem, $H_*(X;\mathbb{k})$ is assumed to be $\mathbb{k}$-free of finite type in each degree and X will be always $\mathbb{k}$-formal: the algebra $S^*(X;\mathbb{k})$ will be linked by quasi-isomorphisms of cochain algebras to $H_*(X;\mathbb{k})$. Therefore

$$HH^*(S^*(X;\mathbb{k});S^*(X;\mathbb{k})^\vee)\cong HH^*(H^*(X;\mathbb{k});H^*(X;\mathbb{k})^\vee). \tag{15}$$

Theorem 16. *For $n>1$ odd, as a Batalin-Vilkovisky algebra,*

$$\begin{gathered}\mathbb{H}_*(LS^n;\mathbb{k})=\mathbb{k}[u_{n-1}]\otimes\Lambda a_{-n},\\ \Delta(u_{n-1}^i\otimes a_{-n})=i(u_{n-1}^{i-1}\otimes 1),\\ \Delta(u_{n-1}^i\otimes 1)=0.\end{gathered}$$

Proof. For the algebra structure, Cohen, Jones and Yan [8] proved that $\mathbb{H}_*(LS^n;\mathbb{Z})=\mathbb{k}[u_{n-1}]\otimes\Lambda a_{-n}$ when $\mathbb{k}=\mathbb{Z}$. Their proof works over any $\mathbb{k}$ (alternatively, using Proposition 9, we could assume that $\mathbb{k}=\mathbb{Z}$). Computing Connes boundary map on $HH^*(H^*(S^n);H_*(S^n))$ (Example 12), we see that Δ on $\mathbb{H}_*(LS^n;\mathbb{k})$ is null in even degree and in degree $-n$, and is an isomorphism in degree -1. Therefore $\Delta(u_{n-1}^i\otimes 1)=0$, $\Delta(1\otimes a_{-n})=0$ and $\Delta(u_{n-1}\otimes a_{-n})=\alpha 1$ where α is invertible in $\mathbb{k}$. Replacing a_{-n} by $\frac{1}{\alpha}a_{-n}$ or u_{n-1} by $\frac{1}{\alpha}u_{n-1}$, we can assume up to isomorphisms that $\Delta(u_{n-1}\otimes a_{-n})=1$. Therefore $\{u_{n-1},a_{-n}\}=1$. Using the Poisson relation (7), $\{u_{n-1}^i,a_{-n}\}=iu_{n-1}^{i-1}$. Therefore $\Delta(u_{n-1}^i\otimes a_{-n})=i(u_{n-1}^{i-1}\otimes 1)$. □

Theorem 17. *For $n\geq 2$ even, there exists a constant $\varepsilon_0\in\mathbb{F}_2$ such that as a Batalin-Vilkovisky algebra,*

$$\begin{aligned}\mathbb{H}_*(LS^n;\mathbb{Z})&=\Lambda b\otimes\frac{\mathbb{Z}[a,v]}{(a^2,ab,2av)}\\ &=\bigoplus_{k=0}^{+\infty}\mathbb{Z}v_{2(n-1)}^k\oplus\bigoplus_{k=0}^{+\infty}\mathbb{Z}b_{-1}v^k\oplus\mathbb{Z}a_{-n}\oplus\bigoplus_{k=1}^{+\infty}\frac{\mathbb{Z}}{2\mathbb{Z}}av^k\end{aligned}$$

with $\forall k\geq 0$, $\Delta(v^k)=0$, $\Delta(av^k)=0$ and

$$\Delta(bv^k)=\begin{cases}(2k+1)v^k+\varepsilon_0 av^{k+1} & \text{if } n=2\\ (2k+1)v^k & \text{if } n\geq 4.\end{cases}$$

Proof. For the algebra structure, Cohen, Jones and Yan [8] proved the equality. Computing Connes boundary map on $HH^*(H^*(S^n); H_*(S^n))$ (Example 12), we see that Δ on $\mathbb{H}_*(LS^n;\mathbb{k})$ is null in even degree and is injective in odd degree.

Case $n \neq 2$: this case is simple, since all the generators of $\mathbb{H}_*(LS^n)$, v^k, bv^k and av^k, $k \geq 0$, have different degrees. Using Example 12, we also see that for all $k \geq 0$,

$$\Delta : \mathbb{H}_{-1+2k(n-1)} = \mathbb{Z}b_{-1}v^k \hookrightarrow \mathbb{H}_{2k(n-1)} = \mathbb{Z}v^k$$

has cokernel isomorphic to $\frac{\mathbb{Z}}{(2k+1)\mathbb{Z}}$. Therefore $\Delta(bv^k) = \pm(2k+1)v^k$. By replacing b_{-1} by $-b_{-1}$, we can assume up to isomorphims that $\Delta(b) = 1$. Let $k \geq 1$. Let $\alpha_k \in \{-2k-1, 2k+1\}$ such that $\Delta(bv^k) = \alpha_k v^k$. Using formula (6), we obtain that $\Delta(bv^k v^k) = (2\alpha_k - 1)v^{2k}$. We know that $\Delta(bv^{2k}) = \pm(4k+1)v^{2k}$. Therefore α_k must be equal to $2k+1$.

Case $n = 2$: this case is complicated, since for $k \geq 0$, v^k and av^{k+1} have the same degree. Using Example 12, we also see that

$$\Delta : \mathbb{H}_{-1+2k} = \mathbb{Z}b_{-1}v^k \hookrightarrow \mathbb{H}_{2k} = \mathbb{Z}v^k \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}av^{k+1}$$

has cokernel, denoted CokerΔ, isomorphic to $\frac{\mathbb{Z}}{(2k+1)\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$. There exists unique $\alpha_k \in \mathbb{Z} - \{0\}$ and $\varepsilon_k \in \frac{\mathbb{Z}}{2\mathbb{Z}}$ such that $\Delta(bv^k) = \alpha_k v^k + \varepsilon_k av^{k+1}$. The injective map Δ fits into the commutative diagram of short exact sequences (Noether's Lemma)

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathbb{H}_{-1+2k} & \xrightarrow{id} & \mathbb{H}_{-1+2k} & \to & 0 & \to & 0 \\
 & & \downarrow{\times 2} & & \downarrow & & \downarrow & & \\
0 & \to & \mathbb{H}_{-1+2k} & \xrightarrow{\Delta} & \mathbb{H}_{2k} & \to & \mathrm{Coker}\Delta & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow\cong & & \\
0 & \to & \frac{\mathbb{Z}}{2\mathbb{Z}} & \xrightarrow{\overline{\Delta}} & \frac{\mathbb{Z}}{2\alpha_k\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}} & \to & \mathrm{Coker}\overline{\Delta} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The cokernel of $\overline{\Delta}$, denoted Coker$\overline{\Delta}$ is of cardinal $2|\alpha_k|$. So $|\alpha_k| = 2k+1$. Therefore $\Delta(bv^k) = \pm(2k+1)v^k + \varepsilon_k av^{k+1}$.

By replacing b_{-1} by $-b_{-1}$, we can assume up to isomorphims that $\Delta(b) = 1 + \varepsilon_0 av$. Using formula (6), we obtain that

$$\Delta(bv^k v^l) = (\alpha_k + \alpha_l - 1)v^{k+l} + (\varepsilon_k + \varepsilon_l - \varepsilon_0)av^{k+l+1}.$$

Therefore

$$\Delta(bv^k v^k) = (2\alpha_k - 1)v^{2k} + \varepsilon_0 av^{2k+1} = \pm(4k+1)v^{2k} + \varepsilon_{2k}av^{2k+1}.$$

So $\alpha_k = 2k+1$, $\varepsilon_{2k} = \varepsilon_0$ and $\varepsilon_{2k+1} = \varepsilon_{2k} + \varepsilon_1 - \varepsilon_0 = \varepsilon_1$.

The map $\Theta : \mathbb{H}_*(LS^2) \to \mathbb{H}_*(LS^2)$ given by $\Theta(b_{-1}v^k) = b_{-1}v^k$, $\Theta(v^k) = v^k + kav^{k+1}$, $\Theta(av^k) = av^k$, $k \geq 0$ is an involutive isomorphism of algebras. Therefore, by replacing v by $v + av^2$, we can assume that $\varepsilon_1 = \varepsilon_0$. So we have proved

$$\Delta(bv^k) = (2k+1)v^k + \varepsilon_0 av^{k+1}, \quad k \geq 0.$$

□

These two cases $\varepsilon_0 = 0$ and $\varepsilon_0 = 1$ correspond to two non-isomorphic Batalin-Vilkovisky algebras whose underlying Gerstenhaber algebras are the same. Therefore even if we have not yet computed the Batalin-Vilkovisky algebra $\mathbb{H}_*(LS^2; \mathbb{Z})$, we have computed its underlying Gerstenhaber algebra. Using the definition of the bracket, straightforward computations give the following corollary.

Corollary 18. *For $n \geq 2$ even, as Gerstenhaber algebra*

$$\mathbb{H}_*(LS^n; \mathbb{Z}) = \Lambda b_{-1} \otimes \frac{\mathbb{Z}[a_{-n}, v_{2(n-1)}]}{(a^2, ab, 2av)}$$

with $\{v^k, v^l\} = 0$, $\{bv^k, v^l\} = -2lv^{k+l}$, $\{bv^k, bv^l\} = 2(k-l)bv^{k+l}$, $\{a, v^l\} = 0$, $\{av^k, bv^l\} = -(2l+1)av^{k+l}$ and $\{av^k, av^l\} = 0$ for all $k, l \geq 0$.

5. When Hochschild cohomology is a Batalin-Vilkovisky algebra

In this section, we recall the structure of Gerstenhaber algebra on the Hochschild cohomology of an algebra whose degrees are bounded. We recall from [26, 22, 27, 20] the Batalin-Vilkovisky algebra on the Hochschild cohomology of the cohomology $H^*(M)$ of a closed oriented manifold M. We compute this Batalin-Vilkovisky algebra $HH^*(H^*(M); H^*(M))$ when M is a sphere.

Through this section, we will work over the prime field $\mathbb{F}_2$. Let A be an augmented graded algebra such that the augmentation ideal $\overline{A}$ is concentrated in degree ≤ -2 and bounded below (or concentrated in degree ≥ 0 and bounded above). Then the (normalized) Hochschild cochain complex, denoted $\mathcal{C}^*(A, A)$, is the complex

$$\mathrm{Hom}(Ts\overline{A}, A) \cong \oplus_{p \geq 0} \mathrm{Hom}((s\overline{A})^{\otimes p}, A)$$

with a differential d_2. For $f \in \mathrm{Hom}((s\overline{A})^{\otimes p}, A)$, the differential $d_2 f \in \mathrm{Hom}((s\overline{A})^{\otimes p+1}, A)$ is given by

$$(d_2 f)([sa_1|\cdots|sa_{p+1}]) := a_1 f([sa_2|\cdots|sa_{p+1}]) \\ + \sum_{i=1}^{p} f([sa_1|\cdots|s(a_i a_{i+1})|\cdots|sa_{p+1}]) + f([sa_1|\cdots|sa_p])a_p$$

The Hochschild cohomology of A with coefficient in A is the homology of the Hochschild cochain complex:

$$HH^*(A;A) := H_*(\mathcal{C}^*(A;A)).$$

We remark that $HH^*(A;A)$ is bigraded. Our degree is sometimes called the total degree: sum of the external degree and the internal degree. The Hochschild cochain complex $\mathcal{C}^*(A,A)$ is a differential graded algebra. For $f \in \mathrm{Hom}((s\overline{A})^{\otimes p}, A)$ and $g \in \mathrm{Hom}((s\overline{A})^{\otimes q}, A)$, the (cup) product of f and g, $f \cup g \in \mathrm{Hom}((s\overline{A})^{\otimes p+q}, A)$ is defined by

$$(f \cup g)([sa_1|\cdots|sa_{p+q}]) := f([sa_1|\cdots|sa_p])g([sa_{p+1}|\cdots|sa_{p+q}]).$$

The Hochschild cochain complex $\mathcal{C}^*(A,A)$ has also a Lie bracket of (lower) degree $+1$.

$$(f\overline{\circ}g)([sa_1|\cdots|sa_{p+q-1}]) := \\ \sum_{i=1}^{p} f\left([sa_1|\cdots|sa_{i-1}|sg([sa_i|\cdots|sa_{i+q-1}])|sa_{i+q}|\cdots|sa_{p+q-1}]\right).$$

$\{f,g\} = f\overline{\circ}g - g\overline{\circ}f$. Our formulas are the same as in the non graded case [13]. We remark that if A is not assumed to be bounded, the formulas are more complicated. Gerstenhaber has shown that $HH^*(A;A)$ equipped with the cup product and the Lie bracket is a Gerstenhaber algebra.

Let M be a closed d-dimensional smooth manifold. Poincaré duality induces an isomorphism of $H^*(M;\mathbb{F}_2)$-modules of (lower) degree d.

$$\Theta : H^*(M;\mathbb{F}_2) \overset{\cap[M]}{\to} H_*(M;\mathbb{F}_2) \cong H^*(M;\mathbb{F}_2)^\vee. \tag{19}$$

More generally, let A be a graded algebra equipped with an isomorphism of A-bimodules of degree d, $\Theta : A \overset{\cong}{\to} A^\vee$. Then we have the isomorphism

$$HH^*(A,\Theta) : HH^*(A,A) \overset{\cong}{\to} HH^*(A,A^\vee).$$

Therefore on $HH^*(A,A)$, we have both a Gerstenhaber algebra structure and an operator Δ given by the dual of Connes boundary map B.

Motivated by the Batalin-Vilkovisky algebra structure of Chas-Sullivan on $\mathbb{H}_*(LM)$, Thomas Tradler [26] proved that $HH^*(A,A)$ is a Batalin-Vilkovisky algebra. See [22, Theorem 1.6] for an explicit proof. In [20] or [27, Corollary 3.4] or [9, Section 1.4] or [19, Theorem B] or [21, Section 11.6], this Batalin-Vilkovisky algebra structure on $HH^*(A,A)$ extends to a structure of algebra on the Hochschild cochain complex $\mathcal{C}^*(A,A)$ over various operads or PROPs: the so-called cyclic Deligne conjecture. Let us compute this Batalin-Vilkovisky algebra structure when M is a sphere.

Proposition 20. *([30] and [31, Corollary 4.2]) Let $d\geq 2$. As Batalin-Vilkovisky algebra, for the Hochschild cohomology of $H^*(S^d;\mathbb{F}_2)=\Lambda x_{-d}$, we have*

$$HH^*(H^*(S^d;\mathbb{F}_2);H^*(S^d;\mathbb{F}_2))\cong \Lambda g_{-d}\otimes\mathbb{F}_2[f_{d-1}]$$

with $\Delta(g_{-d}\otimes f_{d-1}^k)=k(1\otimes f_{d-1}^{k-1})$ and $\Delta(1\otimes f_{d-1}^k)=0, k\geq 0$. In particular, the underlying Gerstenhaber algebra is given by $\{f^k,f^l\}=0$, $\{gf^k,f^l\}=lf^{k+l-1}$ and $\{gf^k,gf^l\}=(k-l)gf^{k+l-1}$ for k, $l\geq 0$.

Proof. Denote by $A:=H^*(S^d;\mathbb{F}_2)$. The differential on $\mathcal{C}^*(A;A)$ is null. Let $f\in\operatorname{Hom}(s\overline{A},A)\subset\mathcal{C}^*(A;A)$ such that $f([sx])=1$. Let $g\in\operatorname{Hom}(\mathbb{F}_2,A)=\operatorname{Hom}((s\overline{A})^{\otimes 0},A)\subset\mathcal{C}^*(A;A)$ such that $g([])=x$. The k-th power of f is the map $f^k\in\operatorname{Hom}((s\overline{A})^{\otimes k},A)$ such that $f^k([sx|\cdots|sx])=1$. The cup product $g\cup f^k\in\operatorname{Hom}((s\overline{A})^{\otimes k},A)$ sends $[sx|\cdots|sx]$ to x. So we have proved that $\mathcal{C}^*(A;A)$ is isomorphic to the tensor product of graded algebras $\Lambda g_{-d}\otimes\mathbb{F}_2[f_{d-1}]$.

The unit 1 and x_{-d} form a linear basis of $H^*(S^d)$. Denote by $1^\vee$ and $x^\vee$ the dual basis of $A^\vee=H^*(S^d)^\vee$. Poincaré duality induces the isomorphism $\Theta: H^*(S^d)\overset{\cong}{\to}H^*(S^d)^\vee$, $1\mapsto x^\vee$ and $x\mapsto 1^\vee$. The two families of elements of the form $1[sx|\cdots|sx]$ and of the form $x[sx|\cdots|sx]$ form a basis of $\mathcal{C}_*(A;A)$. Denote by $1[sx|\cdots|sx]^\vee$ and $x[sx|\cdots|sx]^\vee$ the dual basis in $\mathcal{C}_*(A;A)^\vee$. The isomorphism Θ induces an isomorphism of complexes of degree d, $\widehat{\Theta}:\mathcal{C}^*(A;A)\underset{\cong}{\overset{\mathcal{C}^*(A;\Theta)}{\to}}\mathcal{C}^*(A;A^\vee)\overset{\cong}{\to}\mathcal{C}_*(A;A)^\vee$. Explicitly [22, Section 4] this isomorphism sends $f\in\operatorname{Hom}((s\overline{A})^{\otimes p},A)$ to the linear map $\widehat{\Theta}(f)\in(A\otimes(s\overline{A})^{\otimes p})^\vee\subset\mathcal{C}_*(A;A)^\vee$ defined by

$$\widehat{\Theta}(f)(a_0[sa_1|\cdots|sa_p])=((\Theta\circ f)[sa_1|\cdots|sa_p])\,(a_0).$$

Here with $A=\Lambda x$, $\widehat{\Theta}(f^k)=x[sx|\cdots|sx]^\vee$ and $\widehat{\Theta}(g\cup f^k)=1[sx|\cdots|sx]^\vee$. Computing Connes boundary map $B^\vee$ on $\mathcal{C}_*(A;A)^\vee$ (Example 12) and using that by definition of Δ, $\widehat{\Theta}\circ\Delta=B^\vee\circ\widehat{\Theta}$, we obtain the desired formula for Δ. □

6. The Gerstenhaber algebra $\mathbb{H}_*(LS^2;\mathbb{F}_2)$

Using the same Hochschild homology technique as in section 4, we compute up to an indeterminacy, the Batalin-Vilkovisky algebra $\mathbb{H}_*(LS^2;\mathbb{F}_2)$. Nevertheless, this will give the complete description of the underlying Gerstenhaber algebra on $\mathbb{H}_*(LS^2;\mathbb{F}_2)$.

Lemma 21. *There exist a constant $\varepsilon\in\{0,1\}$ such that as a Batalin-Vilkovisky algebra, the homology of the free loop space on the sphere S^2 is*

$$\mathbb{H}_*(LS^2;\mathbb{F}_2)=\Lambda a_{-2}\otimes\mathbb{F}_2[u_1],$$
$$\Delta(a_{-2}\otimes u_1^k)=k(1\otimes u_1^{k-1}+\varepsilon a_{-2}\otimes u_1^{k+1})\ \textit{and}\ \Delta(1\otimes u_1^k)=0, k\geq 0.$$

Proof. In [8], Cohen, Jones and Yan proved that the Serre spectral sequence for the free loop fibration $\Omega M\overset{j}{\hookrightarrow} LM\overset{ev}{\twoheadrightarrow} M$ is a spectral sequence of algebras converging toward the algebra $\mathbb{H}_*(LM)$. Using Hochschild homology, we see that there is an isomorphism of vector spaces $\mathbb{H}_*(LS^2;\mathbb{F}_2)\cong\mathbb{H}_*(S^2;\mathbb{F}_2)\otimes H_*(\Omega S^2;\mathbb{F}_2)$. Therefore the Serre spectral sequence collapses. Since there is no extension problem, we have the isomorphism of algebras

$$\mathbb{H}_*(LS^2;\mathbb{F}_2)\cong\mathbb{H}_*(S^2;\mathbb{F}_2)\otimes H_*(\Omega S^2;\mathbb{F}_2)=\Lambda(a_{-2})\otimes\mathbb{F}_2[u_1].$$

Computing Connes boundary map on $HH^*(H^*(S^2;\mathbb{F}_2);H_*(S^2;\mathbb{F}_2))$ (Example 12), we see that Δ on $\mathbb{H}_*(LS^2;\mathbb{F}_2)$ is null in even degree and that

$$\Delta:\mathbb{H}_{2k-1}\to\mathbb{H}_{2k}$$

is a linear map of rank 1, $k\geq 0$. In particular Δ is injective in degree -1.

Applying Lemma 11, to the identity map $id:S^2\to S^2$, we see that the composite

$$H_1(\Omega S^2;\mathbb{F}_2)\overset{H_1(j;\mathbb{F}_2)}{\to}H_1(LS^2;\mathbb{F}_2)\overset{\Delta}{\to}H_2(LS^2;\mathbb{F}_2)\overset{H_2(ev;\mathbb{F}_2)}{\to}H_2(S^2;\mathbb{F}_2)$$

is non zero. Since $\mathbb{H}_*(ev)$ is a morphism of algebras, $\mathbb{H}_0(ev)(a_{-2}u_1^2)=0$. And so $\Delta(a_{-2}u_1)=1+\varepsilon a_{-2}u_1^2$ with $\varepsilon\in\mathbb{F}_2$.

We remark that when $b=c$, formula (6) takes the simple form

$$\Delta(ab^2)=\Delta(a)b^2+a\Delta(b^2). \tag{22}$$

Using this formula, we obtain that

$$\Delta(a_{-2}u_1^{2k+1})=\Delta((a_{-2}u_1)(u_1^k)^2)=u_1^{2k}+\varepsilon a_{-2}u_1^{2k+2}\quad k\geq 0.$$

Since $\Delta:\mathbb{H}_1=\mathbb{F}_2a_{-2}u_1^3\oplus\mathbb{F}_2u_1\to\mathbb{H}_2$ is of rank 1 and $\Delta(a_{-2}u_1^3)\neq 0$, $\Delta(u_1)=\lambda\Delta(a_{-2}u_1^3)$ with $\lambda=0$ or $\lambda=1$. Using again formula (22), we have that

$$\Delta(u_1^{2k+1})=\Delta(u_1(u_1^k)^2)=\lambda\Delta(a_{-2}u_1^3)u_1^{2k}=\lambda\Delta(a_{-2}u_1^{2k+3}), k\geq 0.$$

So finally

$$\Delta(a_{-2}u_1^k) = ku_1^{k-1} + \varepsilon k a_{-2}u_1^{k+1} \text{ and } \Delta(u_1^k) = \lambda\Delta(a_{-2}u_1^{k+2}), k \geq 0.$$

The cases $\lambda = 0$ and $\lambda = 1$ correspond to isomorphic Batalin-Vilkovisky algebras: Let $\Theta : \mathbb{H}_*(LS^2;\mathbb{F}_2) \to \mathbb{H}_*(LS^2;\mathbb{F}_2)$ be an automorphism of algebras which is not the identity. Since $\Theta(a_{-2}) \neq 0$, $\Theta(a_{-2}) = a_{-2}$. Since $\Theta(a_{-2})$ and $\Theta(u_1)$ must generate the algebra $\Lambda a_{-2} \otimes \mathbb{F}_2[u_1]$, $\Theta(u_1) \neq a_{-2}u_1^3$. Since $\Theta(u_1) \neq u_1$, $\Theta(u_1) = u_1 + a_{-2}u_1^3$. Therefore there is an unique automorphism of algebras $\Theta : \mathbb{H}_*(LS^2;\mathbb{F}_2) \to \mathbb{H}_*(LS^2;\mathbb{F}_2)$ which is not the identity. Explicitly, Θ is given by $\Theta(u_1^k) = u_1^k + ka_{-2}u_1^{k+2}$, $\Theta(a_{-2}u_1^k) = a_{-2}u_1^k$, $k \geq 0$. One can check that Θ is an involutive isomorphism of Batalin-Vilkovisky algebras who transforms the cases $\lambda = 0$ into the cases $\lambda = 1$ without changing ε. Therefore, by replacing u_1 by $u_1 + a_{-2}u_1^3$, we can assume that $\lambda = 0$. □

Consider the four Batalin-Vilkovisky algebras $\Lambda a_{-2} \otimes \mathbb{F}_2[u_1]$ with $\Delta(a_{-2} \otimes u_1^k) = k(1 \otimes u_1^{k-1} + \varepsilon a_{-2} \otimes u_1^{k+1})$, $\Delta(1 \otimes u_1^k) = \lambda\Delta(a_{-2}u_1^{k+2})$, $k \geq 0$, given by the different values of ε, $\lambda \in \{0, 1\}$. These four Batalin-Vilkovisky algebras have only two underlying Gerstenhaber algebras given by $\{u_1^k, u_1^l\} = 0$, $\{a_{-2}u_1^k, u_1^l\} = lu^{k+l-1} + l(\varepsilon - \lambda)a_{-2}u^{k+l+1}$ and $\{a_{-2}u_1^k, a_{-2}u_1^l\} = (k - l)a_{-2}u^{k+l-1}$ for k, $l \geq 0$. Via the above isomorphism Θ, these two Gerstenhaber algebras are isomorphic.

Corollary 23. *The free loop space modulo 2 homology $\mathbb{H}_*(LS^2;\mathbb{F}_2)$ is isomorphic as Gerstenhaber algebra to the Hochschild cohomology of $H^*(S^2;\mathbb{F}_2)$, $HH^*(H^*(S^2;\mathbb{F}_2);H^*(S^2;\mathbb{F}_2))$.*

7. The Batalin-Vilkovisky algebra $\mathbb{H}_*(LS^2)$

In this section, we complete the calculations of the Batalin-Vilkovisky algebras $\mathbb{H}_*(LS^2;\mathbb{F}_2)$ and $\mathbb{H}_*(LS^2;\mathbb{Z})$ started respectively in sections 6 and 4, using a purely homotopic method.

Theorem 24. *As a Batalin-Vilkovisky algebra, the homology of the free loop space on the sphere S^2 with mod 2 coefficients is*

$$\mathbb{H}_*(LS^2;\mathbb{F}_2) = \Lambda a_{-2} \otimes \mathbb{F}_2[u_1],$$

$$\Delta(a_{-2} \otimes u_1^k) = k(1 \otimes u_1^{k-1} + a_{-2} \otimes u_1^{k+1}) \text{ and } \Delta(1 \otimes u_1^k) = 0, k \geq 0.$$

Theorem 25. *With integer coefficients, as a Batalin-Vilkovisky algebra,*

$$\mathbb{H}_*(LS^2;\mathbb{Z}) = \Lambda b \otimes \frac{\mathbb{Z}[a,v]}{(a^2, ab, 2av)}$$
$$= \bigoplus_{k=0}^{+\infty} \mathbb{Z}v_2^k \oplus \bigoplus_{k=0}^{+\infty} \mathbb{Z}b_{-1}v^k \oplus \mathbb{Z}a_{-2} \oplus \bigoplus_{k=1}^{+\infty} \frac{\mathbb{Z}}{2\mathbb{Z}} av^k$$

with $\forall k \geq 0$, $\Delta(v^k) = 0$, $\Delta(av^k) = 0$ *and* $\Delta(bv^k) = (2k+1)v^k + av^{k+1}$.

Denote by $s : X \hookrightarrow LX$ the trivial section of the evaluation map $ev : LX \twoheadrightarrow X$.

Lemma 26. *The image of* $\Delta : H_1(LS^2;\mathbb{F}_2) \to H_2(LS^2;\mathbb{F}_2)$ *is not contained in the image of* $H_2(s;\mathbb{F}_2) : H_2(S^2;\mathbb{F}_2) \hookrightarrow H_2(LS^2;\mathbb{F}_2)$.

Lemma 27. *The image of* $\Delta : H_1(LS^2;\mathbb{Z}) \to H_2(LS^2;\mathbb{Z})$ *is not contained in the image of* $H_2(s;\mathbb{Z}) : H_2(S^2;\mathbb{Z}) \hookrightarrow H_2(LS^2;\mathbb{Z})$.

Proof of Lemma 27 assuming Lemma 26. Consider the commutative diagram

$$\begin{array}{ccc}
H_1(LS^2;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_2 & \xrightarrow{\cong} & H_1(LS^2;\mathbb{F}_2) \\
\downarrow{\scriptstyle \Delta\otimes_{\mathbb{Z}}\mathbb{F}_2} & & \downarrow{\scriptstyle \Delta} \\
H_2(LS^2;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_2 & \xrightarrow{\cong} & H_2(LS^2;\mathbb{F}_2) \\
\uparrow{\scriptstyle H_2(s;\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{F}_2} & & \uparrow{\scriptstyle H_2(s;\mathbb{F}_2)} \\
H_2(S^2;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_2 & \xrightarrow{\cong} & H_2(S^2;\mathbb{F}_2)
\end{array}$$

Since $H_1(LS^2;\mathbb{Z}) \cong H_0(LS^2;\mathbb{Z}) \cong \mathbb{Z}$, the horizontal arrows are isomorphisms by the universal coefficient theorem. The top rectangle commutes according to Lemma 4.

Suppose that the image of $\Delta : H_1(LS^2;\mathbb{Z}) \to H_2(LS^2;\mathbb{Z})$ is included in the image of $H_2(s;\mathbb{Z})$. Then the image of $\Delta \otimes_{\mathbb{Z}} \mathbb{F}_2$ is included in the image of $H_2(s;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_2$. Using the above diagram, the image of $\Delta : H_1(LS^2;\mathbb{F}_2) \to H_2(LS^2;\mathbb{F}_2)$ is included in the image of $H_2(s;\mathbb{F}_2)$. This contradicts Lemma 26. □

Proof of Theorem 24 assuming Lemma 26. It suffices to show that the constant ε in Lemma 21 is not zero. Suppose that $\varepsilon = 0$. Then by Lemma 21, $\Delta(a_{-2} \otimes u_1) = 1$.

It is well known that $\mathbb{H}_*(s) : \mathbb{H}_*(M) \to \mathbb{H}_*(LM)$ is a morphism of algebras. In particular, let $[S^2]$ be the fundamental class of S^2, $H_2(s)([S^2])$ is the unit of $\mathbb{H}_*(LS^2)$. So $\Delta(a_{-2} \otimes u_1) = H_2(s)([S^2])$. This contradicts Lemma 26. □

The proof of Theorem 25 assuming Lemma 27 is the same. To complete the computation of this Batalin-Vilkovisky algebra on the homology of the free loop space of a manifold, we will relate it to another structure of Batalin-Vilkovisky algebra that arises in algebraic topology: the homology of the double loop space.

Let X be a pointed topological space. The circle S^1 acts on the sphere S^2 by "rotating the earth". Therefore the circle also acts on $\Omega^2 X = map((S^2, \text{North pole}), (X, *))$. So we have a induced operator $\Delta : H_*(\Omega^2 X) \to H_{*+1}(\Omega^2 X)$. With Theorem 32 and the following Proposition, we will able to prove Lemma 26.

Proposition 28. *Let X be a pointed topological space. There is a natural morphism $r : L\Omega X \to map_*(S^2, X)$ of S^1-spaces between the free loop space on the pointed loops of X and the double pointed loop space of X such that:*

- *If we identify S^2 and $S^1 \wedge S^1$, r is a retract up to homotopy of the inclusion $j : \Omega(\Omega X) \hookrightarrow L(\Omega X)$,*
- *The composite $r \circ s : \Omega X \hookrightarrow L(\Omega X) \to map_*(S^2, X)$ is homotopically trivial.*

Proof. Let $\sigma : S^2 \twoheadrightarrow \frac{S^1 \times S^1}{S^1 \times *} = S^1_+ \wedge S^1$ be the quotient map that identifies the North pole and the South pole on the earth S^2. The circle S^1 acts without moving the based point on $S^1_+ \wedge S^1$ by multiplication on the first factor. On the torus $S^1 \times S^1$, the circle can act by multiplication on both factors. But when you pinch a circle to a point in the torus, the circle can act only on one factor. If we make a picture, we easily see that $\sigma : S^2 \twoheadrightarrow S^1_+ \wedge S^1$ is compatible with the actions of S^1. Therefore $r := map_*(\sigma, X) : L\Omega X \to map_*(S^2, X)$ is a morphism of S^1-spaces.

- Let $\pi : S^1_+ \wedge S^1 \twoheadrightarrow S^1 \wedge S^1 = \frac{S^1_+ \wedge S^1}{* \times S^1}$ be the quotient map. The inclusion map $j : \Omega(\Omega X) \to L(\Omega X)$ is $map_*(\pi, X)$. The composite $\pi \circ \sigma : S^2 \twoheadrightarrow S^1 \wedge S^1$ is the quotient map obtained by identifying a meridian with a point in the sphere S^2. The composite $\pi \circ \sigma$ can also be viewed as the quotient map from the non reduced suspension of S^1 to the reduced suspension of S^1. So the composite $\pi \circ \sigma : S^2 \twoheadrightarrow S^1 \wedge S^1$ is a homotopy equivalence. Let $\Theta : S^1 \wedge S^1 \overset{\cong}{\to} S^2$ be any given homeomorphism. The composite $\Theta \circ \pi \circ \sigma : S^2 \to S^2$ is of degree ± 1. The reflection through the equatorial plane is a morphism of S^1-spaces. By replacing eventually σ by its composite with the previous reflection, we can suppose that $\Theta \circ \pi \circ \sigma : S^2 \to S^2$ is homotopic to the identity map of S^2, i. e. $\sigma \circ \Theta$ is a section of π up to homotopy. Therefore $map_*(\sigma \circ \Theta, X) = map_*(\Theta, X) \circ r$ is a retract of j up to homotopy.

• Let $\rho : S^1_+ \wedge S^1 = \frac{S^1\times S^1}{S^1\times *} \twoheadrightarrow S^1$ be the map induced by the projection on the second factor. Since $\pi_2(S^1) = *$, the composite $\rho\circ\sigma$ is homotopically trivial. Therefore $r\circ s$, the composite of $r = map_*(\sigma, X)$ and $s = map_*(\rho, X) : \Omega X \to L(\Omega X)$ is also homotopically trivial. □

Proof of Lemma 26. Denote by $ad_{S^n} : S^n \to \Omega S^{n+1}$ the adjoint of the identity map $id : S^{n+1} \to S^{n+1}$. The map $L(ad_{S^2}) : LS^2 \to L\Omega S^3$ is obviously a morphism of S^1-spaces. Therefore using Proposition 28, the composite $r\circ L(ad_{S^2}) : LS^2 \to L\Omega S^3 \to \Omega^2S^3$ is also a morphism of S^1-spaces. Therefore $H_*(r\circ L(ad_{S^2}))$ commutes with the corresponding operators Δ in $H_*(LS^2)$ and $H_*(\Omega^2S^3)$.

Consider the commutative diagram up to homotopy

(29)
$$\begin{array}{ccccc}
\Omega S^2 & \xrightarrow{j} & LS^2 & \xleftarrow{s} & S^2 \\
\downarrow{\scriptstyle \Omega(ad_{S^2})} & & \downarrow{\scriptstyle L(ad_{S^2})} & & \downarrow{\scriptstyle ad_{S^2}} \\
\Omega^2 S^3 & \xrightarrow{j} & L\Omega S^3 & \xleftarrow{s} & \Omega S^3 \\
& {\scriptstyle id}\searrow & \downarrow{\scriptstyle r} & \swarrow{\scriptstyle *} & \\
& & \Omega^2 S^3 & &
\end{array}$$

Using the left part of this diagram, we see that $\pi_1(r\circ L(ad))$ maps the generator of $\pi_1(LS^2) = \mathbb{Z}(j\circ ad_{S^1})$ to the composite $\Omega(ad_{S^2})\circ ad_{S^1} : S^1 \to \Omega S^2 \to \Omega^2S^3$ which is the generator of $\pi_1(\Omega^2S^3) \cong \mathbb{Z}$. Therefore $\pi_1(r\circ L(ad))$ is an isomorphism.

So we have the commutative diagram

$$\begin{array}{ccccc}
\pi_1(LS^2)\otimes\mathbb{F}_2 & \xrightarrow[\cong]{hur} & H_1(LS^2;\mathbb{F}_2) & \xrightarrow{\Delta} & H_2(LS^2;\mathbb{F}_2) \\
\downarrow{\scriptstyle \pi_1(r\circ L(ad_{S^2}))\otimes\mathbb{F}_2}\,\cong & & \downarrow{\scriptstyle H_1(r\circ L(ad_{S^2});\mathbb{F}_2)} & & \downarrow{\scriptstyle H_2(r\circ L(ad_{S^2});\mathbb{F}_2)} \\
\pi_1(\Omega^2S^3)\otimes\mathbb{F}_2 & \xrightarrow[\cong]{hur} & H_1(\Omega^2S^3;\mathbb{F}_2) & \xrightarrow{\Delta} & H_2(\Omega^2S^3;\mathbb{F}_2)
\end{array}$$

By Theorem 32, $\Delta : H_1(\Omega^2S^3;\mathbb{F}_2) \to H_2(\Omega^2S^3;\mathbb{F}_2)$ is non zero. Therefore using the above diagram, the composite $H_2(r\circ L(ad_{S^2}))\circ\Delta$ is also non zero. On the other hand, using the right part of diagram (29), we have that the composite $H_2(r\circ L(ad_{S^2}))\circ H_2(s)$ is null. □

Corollary 30. *The free loop space modulo 2 homology $\mathbb{H}_*(LS^2;\mathbb{F}_2)$ is not isomorphic as Batalin-Vilkovisky algebra to the Hochschild cohomology of $H^*(S^2;\mathbb{F}_2)$, $HH^*(H^*(S^2;\mathbb{F}_2);H^*(S^2;\mathbb{F}_2))$.*

This means exactly that there exists no isomorphism between $\mathbb{H}_*(LS^2;\mathbb{F}_2)$ and $HH^*(H^*(S^2;\mathbb{F}_2);H^*(S^2;\mathbb{F}_2))$ which at the same time,

- is an isomorphism of algebras and
- commutes with the Δ operators,

although separately

- there exists an isomorphism of algebras between $\mathbb{H}_*(LS^2;\mathbb{F}_2)$ and $HH^*(H^*(S^2;\mathbb{F}_2);H^*(S^2;\mathbb{F}_2))$ (Corollary 23) and
- there exists also an isomorphism commuting with the Δ operators between them.

Proof. By Proposition 20, $HH^*(H^*(S^2);H^*(S^2))$ is the Batalin-Vilkovisky algebra given by $\varepsilon=0$ in Lemma 21. On the contrary, by Theorem 24, $\mathbb{H}_*(LS^2;\mathbb{F}_2)$ is the Batalin-Vilkovisky algebra given by $\varepsilon=1$. At the end of the proof of Lemma 21, we saw that the two cases $\varepsilon=0$ and $\varepsilon=1$ correspond to two non isomorphic Batalin-Vilkovisky algebras. ☐

More generally, we believe that for any prime p, the free loop space modulo p of the complex projective space $\mathbb{H}_*(L\mathbb{CP}^{p-1};\mathbb{F}_p)$[2] is not isomorphic as Batalin-Vilkovisky algebra to the Hochschild cohomology $HH^*(H^*(\mathbb{CP}^{p-1};\mathbb{F}_p);H^*(\mathbb{CP}^{p-1};\mathbb{F}_p))$. Such phenomena for formal manifolds should not appear over a field of characteric 0.

Recall that by Poincaré duality, we have the isomorphism

$$(19)\quad \Theta : H^*(S^2)\overset{\cong}{\to} H^*(S^2)^\vee.$$

Therefore we have the isomorphism

$$HH^*(H^*(S^2);\Theta) : HH^*(H^*(S^2);H^*(S^2))\overset{\cong}{\to} HH^*(H^*(S^2);H^*(S^2)^\vee).$$

Consider any isomorphism of graded algebras

$$\mathbb{H}_*(LS^2)\cong HH^*(S^*(S^2);S^*(S^2)). \tag{31}$$

By Corollary 23, such isomorphism exists. Cohen and Jones ([7, Theorem 3] and [5]) proved that such isomorphism exists for any manifold M. Since S^2 is formal, we have the isomorphism of algebras

$$(2)\quad HH^*(S^*(S^2);S^*(S^2))\overset{\cong}{\to} HH^*(H^*(S^2);H^*(S^2)).$$

By [17], we have the isomorphisms of $H_*(S^1)$-modules

$$H_*(LS^2)\overset{(14)}{\cong} HH^*(S^*(S^2);S^*(S^2)^\vee)\overset{(15)}{\cong} HH^*(H^*(S^2);H^*(S^2)^\vee).$$

Corollary 30 implies that the following diagram does not commute over $\mathbb{F}_2$:

[2] Bökstedt and Ottosen [1] have recently announced the computation of Batalin-Vilkovisky algebra $\mathbb{H}_*(L\mathbb{CP}^n;\mathbb{F}_p)$.

$$\begin{array}{ccccc}
 & & HH^*(S^*(S^2);S^*(S^2)^\vee) & \xrightarrow{(15)} & HH^*(H^*(S^2);H^*(S^2)^\vee) \\
 & \overset{(14)}{\nearrow} & & & \uparrow {\scriptstyle HH^*(H^*(S^2);\Theta)} \\
H_*(LS^2) & & & & \\
 & \underset{(31)}{\searrow} & & & \\
 & & HH^*(S^*(S^2);S^*(S^2)) & \xrightarrow{(2)} & HH^*(H^*(S^2);H^*(S^2))
\end{array}$$

This is surprising because as explained by Cohen and Jones [7, p. 792], the composite of the isomorphism (14) given by Jones in [17] and an isomorphism induced by Poincaré duality should give an isomorphism of algebras between $\mathbb{H}_*(LS^2)$ and $HH^*(S^*(S^2);S^*(S^2))$.

8. Appendix by Gerald Gaudens and Luc Menichi.

Let X be a pointed topological space. Recall that the circle S^1 acts on the double loop space $\Omega^2 X$. Consider the induced operator $\Delta : H_*(\Omega^2 X) \to H_{*+1}(\Omega^2 X)$. Getzler [14] has shown that $H_*(\Omega^2 X)$ equipped with the Pontryagin product and this operator Δ forms a Batalin-Vilkovisky algebra. In [12], Gerald Gaudens and the author have determined this Batalin-Vilkovisky algebra $H_*(\Omega^2 S^3;\mathbb{F}_2)$. The key was the following Theorem. In [18, Proposition 7.46], answering to a question of Gerald Gaudens, Sadok Kallel and Paolo Salvatore give another proof of this Theorem.

Theorem 32. [12] *The operator* $\Delta : H_1(\Omega^2 S^3;\mathbb{F}_2) \to H_2(\Omega^2 S^3;\mathbb{F}_2)$ *is non trivial.*

Both proofs [12] and [18, Proposition 7.46] are unpublished and publicly unavailable yet. So the goal of this section is to give a proof of this theorem which is as simple as possible.

Denote by $*$ the Pontryagin product in $H_*(\Omega^2 X)$ and by $\circ$ the map induced in homology by the composition map $\Omega^2 X \times \Omega^2 S^2 \to \Omega^2 X$. Denote by $\Omega^2_n S^2$, the path-connected component of the degree n maps. Denote by v_1 the generator of $H_1(\Omega^2_0 S^2;\mathbb{F}_2)$ and by $[1]$ the generator of $H_0(\Omega^2_1 S^2;\mathbb{F}_2)$.

Lemma 33. *For* $x \in H_*(\Omega^2 X;\mathbb{F}_2)$, $\Delta x = x \circ (v_1 * [1])$.

Proof. The circle S^1 acts on the sphere S^2. Therefore we have a morphism of topological monoids $\Theta : (S^1, 1) \to (\Omega^2_1 S^2, id_{S^2})$. The action of S^1 on $\Omega^2 X$ is the composite $S^1 \times \Omega^2 X \overset{\Theta\times\Omega^2 X}{\to} \Omega^2_1 S^2 \times \Omega^2 X \overset{\circ}{\to} \Omega^2 X$. Therefore for $x \in H_*(\Omega^2 X;\mathbb{F}_2)$, $\Delta x = x \circ (H_1(\Theta)[S^1])$.

Suppose that $H_1(\Theta)[S^1] = 0$. Then for any topological space X, the operator Δ on $H_*(\Omega^2 X;\mathbb{F}_2)$ is null. Therefore, for any x and $y \in$

$H_*(\Omega^2 X;\mathbb{F}_2)$, $\{x,y\}=\Delta(xy)-(\Delta x)y-x(\Delta y)=0$. That is the modulo 2 Browder brackets on any double loop space are null. This is obviously false. For example, Cohen in [3] explains that the Gerstenhaber algebra $H_*(\Omega^2\Sigma^2 Y)$ has in general many non trivial Browder brackets. So the assumption $H_1(\Theta)[S^1]=0$ is false.

Since the loop multiplication by id_{S^2} in the H-group $\Omega^2 S^2$, is a homotopy equivalence, the Pontryagin product by [1], $*[1]: H_*(\Omega^2_0 S^2)\overset{\cong}{\to} H_*(\Omega^2_1 S^2)$ is an isomorphism. Therefore $v_1*[1]$ is a generator of $H_1(\Omega^2_1 S^2)$ So $H_1(\Theta)[S^1]=v_1*[1]$. So finally

$$\Delta x = x\circ (H_1(\Theta)[S^1]) = x\circ (v_1 * [1]).$$

□

Recall that v_1 denotes the generator of $H_1(\Omega^2_0 S^2;\mathbb{F}_2)$.

Lemma 34. *In the Batalin-Vilkovisky algebra $H_*(\Omega^2 S^2;\mathbb{F}_2)$, $\Delta(v_1)=v_1*v_1$.*

Proof. Recall that [1] is the generator of $H_0(\Omega^2_1 S^2)$. By Lemma 33,

$$\Delta[1]=[1]\circ(v_1*[1])=(v_1*[1]).$$

Denote by $Q: H_q(\Omega^2_n S^2)\to H_{2q+1}(\Omega^2_{2n}S^2)$ the Dyer-Lashof operation. It is well known that $Q[1]=v_1*[2]$. So by [4, Theorem 1.3 (4) p. 218]

$$\{v_1*[2],[1]\}=\{Q[1],[1]\}=\{[1],\{[1],[1]\}\}.$$

By [4, Theorem 1.2 (3) p. 215], $\{[1],[1]\}=0$. Therefore on one hand, $\{v_1*[2],[1]\}$ is null. And on the other hand, using the Poisson relation (7), since $\{[2],[1]\}=\{[1]*[1],[1]\}=2\{[1],[1]\}*[1]=0$,

$$\{v_1*[2],[1]\}=\{v_1,[1]\}*[2]+v_1*\{[2],[1]\}=\{v_1,[1]\}*[2].$$

Since $*[1]: H_*(\Omega^2 S^2)\overset{\cong}{\to} H_*(\Omega^2 S^2)$ is an isomorphism, we obtain that Browder bracket $\{v_1,[1]\}$ is null. Therefore,

$$\Delta(v_1*[1])=(\Delta v_1)*[1]+v_1*(\Delta[1])=((\Delta v_1)-v_1*v_1)*[1].$$

But $\Delta(v_1*[1])=(\Delta\circ\Delta)([1])=0$. Therefore (Δv_1) must be equal to v_1*v_1. □

Proof of Theorem 32. We remark that since Δ preserves path-connected components and since the loop multiplication of two homotopically trivial loops is a homotopically trivial loop, $H_*(\Omega^2_0 S^2)$ is a sub Batalin-Vilkovisky algebra of $H_*(\Omega^2 S^2)$.

Let $S^1\hookrightarrow S^3\overset{\eta}{\twoheadrightarrow} S^2$ be the Hopf fibration. After double looping, the Hopf fibration gives the fibration $\Omega^2 S^1\hookrightarrow \Omega^2 S^3\overset{\Omega^2\eta}{\twoheadrightarrow}\Omega^2_0 S^2$ with contractile fiber $\Omega^2 S^1$ and path-connected base $\Omega^2_0 S^2$. Therefore

$\Omega^2\eta : \Omega^2 S^3 \overset{\simeq}{\to} \Omega^2_0 S^2$ is a homotopy equivalence. And so $H_*(\Omega^2\eta) : H_*(\Omega^2 S^3) \overset{\cong}{\to} H_*(\Omega^2_0 S^2)$ is an isomorphism of Batalin-Vilkovisky algebras.

Let u_1 be the generator of $H_1(\Omega^2 S^3)$. Lemma 34 implies that $\Delta(u_1) = u_1 * u_1$. Since $u_1 * u_1$ is non zero in $H_*(\Omega^2 S^3; \mathbb{F}_2)$, $\Delta(u_1)$ is non trivial. □

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UMR 6093 associée au CNRS, Université d'Angers, Faculté des Sciences, 2 Boulevard Lavoisier, 49045 Angers, FRANCE
E-mail address: `firstname.lastname at univ-angers.fr`