

TOPOLOGICAL FIELD THEORY ON THE HOCHSCHILD HOMOLOGY OF A CALABI-YAU ALGEBRA

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- PLAN:
- I. HOCHSCHILD (CO)HOMOLOGY AND STRING TOPOLOGY
 - II. BABY ^{AND} BLUE THEOREMS (BABY = BV PART
BLUE = AS IN TITLE)
 - III. PROOF OF THE BABY THEOREM
 - IV. EXAMPLES OF STRING TOPOLOGY
 - V. HYPOTHESIS OF BABY/BUE THEOREM, IN LURIE'S LANGUAGE, MEANS A IS A CALABI-YAU ALGEBRA.

FIELD (IN PARTICULAR OF CHARACTER $p \neq 0$ — COULD DO SOMETHING EASIER IN CHARACTER 0, VIA RATIONAL HOMOTOPY THEORY) OR COSTRUCO

I. LET A BE A dga, M AN (A, A) -BIMODULE
↳ EVERYTHING DIFF GRADED HERE

Def: HOCHSCHILD COHOMOLOGY

$$HH^*(A, M) = \text{Ext}_{A \otimes A}^*(A, M)$$

EXPLICITLY, LET $\mathcal{E}_A = B(A, A, A) \xrightarrow{\simeq} A$ QUASI-ISO OF A -BIMOD

$$C^*(A, M) = \text{Hom}_{A\text{-BIMOD}}(B(A, A, A), M) \xrightarrow{H_*} HH^*(A, M)$$

$$C_*^*(A, M) = M \otimes_{A \otimes A^{\text{op}}} B(A, A, A) \longrightarrow HH_*(A, M)$$

LET $B: HH_*(A, A) \rightarrow HH_{*+1}(A, A)$ THE COMPLEX BOUNDARY MAP.
 $B^2 = 0$

NOTE: $HH^*(A, A^V) = HH_*(A, A)^V \leftarrow$ DUAL VECTOR SPACE

$\hookrightarrow B^V: HH^*(A, A^V) \rightarrow HH^{*-1}(A, A^V)$

THM (GROSTENHABER) $HH^*(A, A)$ IS A GROSTENHABER ALG,
 I.E. IT'S A GRADED COMMUTATIVE ALG EQUIPPED WITH A
 LIE BRACKET OF DEGREE $\begin{bmatrix} -1 \\ +1 \end{bmatrix}$ $\{, \}$ SATISFYING $\{a, bc\} = \{a, b\}c \pm b\{a, c\}$

CONVENTION: $A_n = A^{-n}$
 GRAD. STRUCT ON A_* , $\{, \}$ OF DEG $\neq 1$
 $A^* \text{ --- } -1$

DEF: A BV-ALG IS A GROSTENHABER ALG EQUIPPED WITH
 AN OPERATOR $\Delta: A_i \rightarrow A_{i+1}$ S.T. $\Delta^2 = 0$, AND
 SUCH THAT THE LIE BRACKET IS GIVEN BY
 $\{a, b\} = \pm \Delta(ab) \mp (\Delta a)b \mp a(\Delta b)$

EX. M^d d-DIMENSIONAL, SIMPLY CONNECTED, COMPACT,
 CLOSED MFD

THM (CHAS-SULLIVAN) $H_{*+d}(LM) \cong H_*(LM)$ IS A BV-ALG.

CONJECTURE: THERE IS AN \mathbb{R} BO OF GROSTENHABER ALGEBRAS

$HH^*(S^*M, S^*M) \cong H_*(LM)$
 SINGULAR COCHAIN OF M

PROVED OVER π OF FEUX-THOMAS TIME !!

NOTE: THERE IS A FUNCTORIALITY ON BOTH SIDES
WRT QUASI-ISOS / HOMOTOPY EQUIV.

FUN CONJECTURE PROVED BY MALM FOR $C_*(SM)$
INSTEAD OF $S^*(M)$ — DON'T NEED M SIMPLY-CONN.
WHICH IMPLIES THE OTHER CASE BY MENICHI-THOMAS
($S^*(M)$)

STRATEGIES

- ① DEFINE A BV-ALGEBRA STRUCTURE ON $HH^*(S^*(M), S^*(M))$
- ② GIVE AN ISO OF ALGEBRAS COMMUTING WITH THE Δ -OPERATORS.

II BABY THEOREM [M] (MIXTURE OF FEUX-THOMAS + GINZBURG)

LET A BE A DIFF GRADED ALGEBRA. $\mathbb{F}[M]$ BE
A CLASS IN $HH^{-d}(A, A^v)$. LET $\eta: \mathbb{F} \rightarrow A$ UNIT MAP

$$\eta^* = HH^*(\eta, A): HH^*(A, A^v) \rightarrow HH^*(\mathbb{F}, A^v) = H(A^v)$$

$$[m] \mapsto \eta^*[m]$$

SUPPOSE THAT

a) THE MORPHISM OF LEFT $H(A)$ -MODULES (USING A^v IS AN A -MODULE)

$$H(A) \longrightarrow H(A^v)$$

$$a \longmapsto a \cdot \eta^*[m]$$

IS AN ISOMORPHISM

THEN

1) ~~THE~~ THERE IS AN ISO OF VECTOR SPACES

$$HH^*(A, A) \xleftrightarrow{\cong} HH^*(A, A^v)$$

$$a \longmapsto a \cap [m]$$

ASSUMING ALSO

b) $B^V([m]) = 0$

THEN 2) THE GOLTENHABER ALG $HH^*(A, A)$ EQUIPPED WITH $\Delta := \mathcal{D} \circ B^V \circ \mathcal{D}^{-1}$ IS A BV-ALGEBRA.

BLUE THEOREM [LURIE EX 4.2.8]

LET S BE A GOOD SYMMETRIC MONOIDAL $(\infty, 1)$ -CAT (FOR LUC, $S =$ CATEGORY OF CHAIN COMPLEXES)

LET A BE AN (A, \circ) -ALGEBRA A^V IN S EQUIPPED WITH A MAP OF COMPLEXES OF DEGREE d

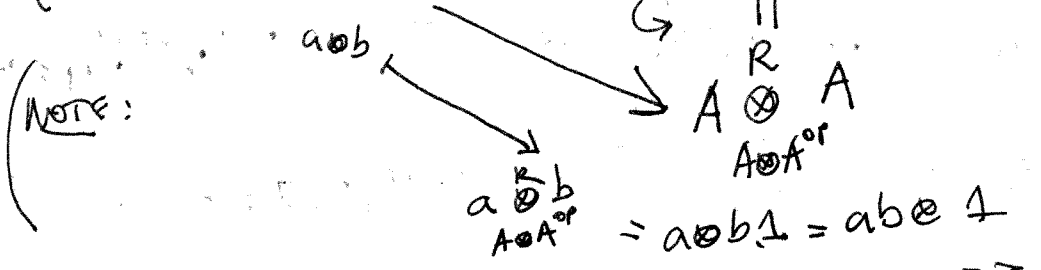
$$A \otimes_{A \otimes A^{op}} B(A, A, A) \xrightarrow{tr} \mathbb{F} \quad (tr = m \text{ ABOVE})$$

SUCH THAT $B^V(tr) = tr \circ B$ IS CHAIN HOMOGENEOUS TO THE ZERO MAP (LURIE "SO(2)-EQUIVARIANT") (\Leftrightarrow CONDITION b) ABOVE)

AND SUCH THAT THE ADJOINT OF THE COMPOSITE

$$A \otimes A \xrightarrow{m} A \xrightarrow{C_*(\eta, A)} C_*(A, A) \xrightarrow{tr} \mathbb{F}$$

IS THE ADJOINT OF A QUASI-ISOMORPHISM $A \xrightarrow{\sim} A^V$ (\Leftrightarrow CONDITION a), NON-DEGENERATE PAIRING IN H_*)



THEN BY THE COORDIN HYPOTHESIS, NON-COMPACT POSITIVE-BORN VERSION 4.2.11, THERE IS A SYMMETRIC MONOIDAL

$\mathcal{Z} : \text{Bord}_2 \longrightarrow \text{Alg}(A) \quad (\text{ALGEBRAS IN } \mathcal{Z})$

SUCH THAT $\mathcal{Z}(+) = A$.

IN PARTICULAR, $\mathcal{Z}(S^1) = A \otimes_{A \otimes A^{\text{op}}} B(A, A, A)$ IS A

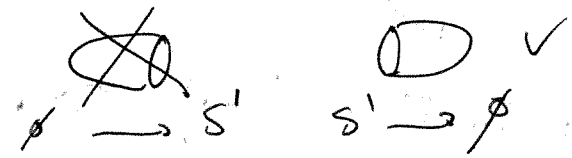
NON-COMPACT CLOSED TOPOLOGICAL CONFORMAL FIELD THEORY.

^(RED) DEFINITION 4.2.10 $\text{Bord}_2^{\text{OR, NC}}$ IS A 2-CATEGORY

Obj = ORIENTED 0-MFDS $\begin{matrix} + \\ - \end{matrix}$

1-MORPH = 1-di ORIENTED COBORDISMS $\begin{matrix} + \\ - \end{matrix} \bigcup \begin{matrix} + \\ - \end{matrix}$

2-MORPH = 2-di ORIENTED COBORDISM SUCH THAT EVERY CONNECTED COMPONENT HAS AN INCOMING BDRY PART. IN PARTICULAR



$\rightarrow HH_*(A, A)$ IS A NON-UNITAL BV-ALG AND

$HH^*(A, A^v)$ IS A UNITAL BV-ALG.

THEREFORE, THE BLUE THEOREM IMPLIES PART 2) OF THE BABY THEOREM, RESTRICTING TO GENUS 0 AND TAKING HOMOLOGY. MORE GENERALLY, $HH_*(A, A)$ IS A HOMOLOGICAL CFT, i.e. AN ALGEBRA OVER THE PROP $\{H_*(M_{g, p+q})\}_{p \geq 1}$

(TAKING DUALS GET AN ACTION OF $\{H_*(M_{g, p+q})\}_{p \geq 1}$.)

(HERE WE USE $H_*(\mathbb{R}D_2) \cong_{\text{OPERAD}} \{H_*(M_{0, k, 1})\}$.)

CONJECTURE : $\mathcal{C}^*(A, A^V)$ IS AN ALGEBRA OVER $S_*(\mathbb{F}D_2)$ (DERIVED CYCLIC DERIVED CONJECTURE)

(NOT TO TALK ABOUT $S_*(Mg, ptg) \dots$)

SKETCH OF PROOF OF BABY THEOREM :

(LET $B(A, A, A)$ ^{DOUBLE} ~~BAR~~ ^{RETRACTION} $\xrightarrow{\cong} A$ HAVE $[m] \in HH^{-d}(A, A^V)$
 $HH(\text{Hom}_{A\text{-bimod}}(B(A, A, A), A^V))$)

$[m]$ IS REPRESENTED BY A MORPHISM OF A -BIMODULES OF DEGREE $-d$ COMMUTING WITH DIFFERENTIALS

$$m: B(A, A, A) \longrightarrow A^V$$

a) MEANS EXACTLY THAT m IS A QUASI-ISOMORPHISM.

WE HAVE A SEQUENCE OF QUASI-ISO OF A -BIMODULES.

$$A \xleftarrow[\varepsilon_A]{\cong} B(A, A, A) \xrightarrow[m]{\cong} A^V$$

THIS MEANS THAT $A \cong A^V$ IN THE DERIVED CATEGORY OF (A, A) -BIMODULES.

BY APPLYING $HH^*(A, -)$, OBTAIN AN ISO AS ^{GRADED} VECTOR SPACES

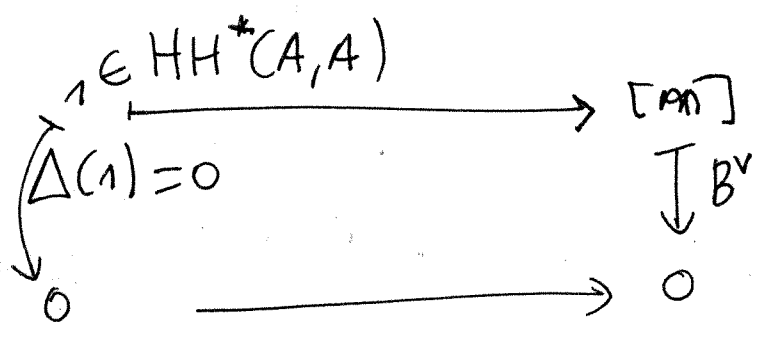
$$HH^*(A, A) \cong HH^*(A, A^V)$$

ONE CAN SHOW THAT THERE IS A BV-STRUCTURE \leftarrow EXTENDING THE EXISTING PARTIAL STRUCTURE ON B SIDES

$$\text{IFF } B^V([m]) = 0$$

Ginzburg: $HH^*(A, A) \xrightarrow{\cong} HH_*(A, A)$
 ASSUMES ISO CLOSELY RELATED

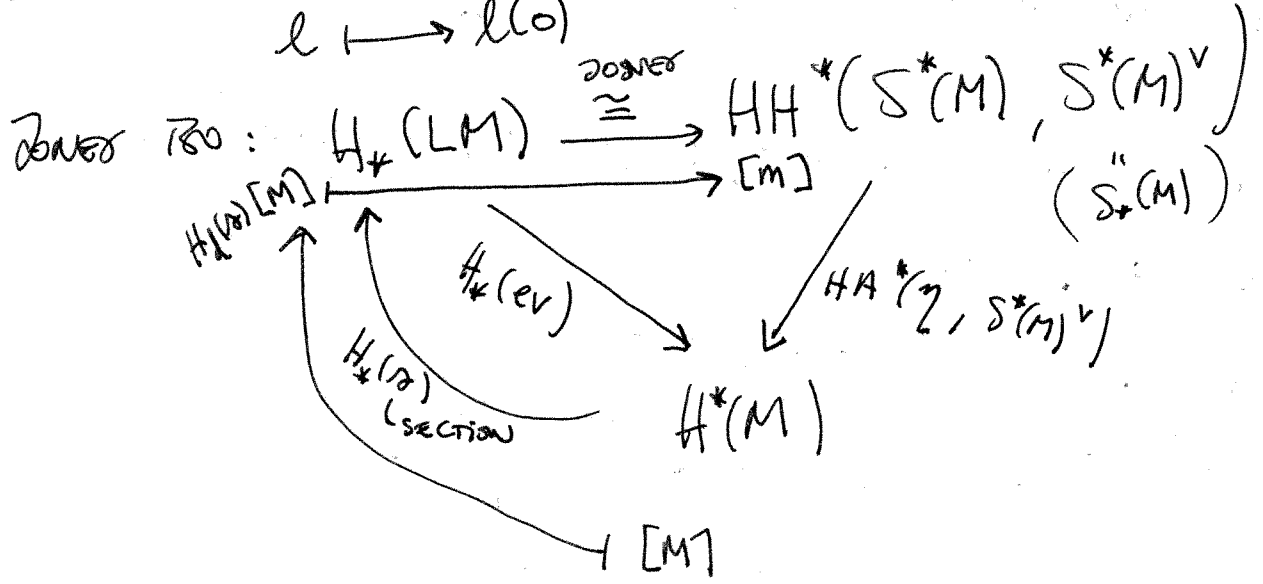
THE CONDITION COMES FROM



IV EXAMPLE OF STRING TOPOLOGY [LURIE/MENICHI]

WE WANT TO TAKE $A = S^*(M)$ IN THE BABY/BWE THM

LET $ev: LM \rightarrow M$
 $l \mapsto l(0)$



CLAIM: $[M]$ SATISFIES HYPOTHESIS of THM

PF: $HH^*(\eta, S^*(M)^v)([M]) = [M]$ BECAUSE DIAGRAM COMMUTES

AND $[M]$ GIVES APPROPRIATE ISO BY PD..

→ SO a) ✓

For b), $B^V(m) = B^V \circ \text{Jones} \circ H_*(S^1 | M)$
 $= \text{Jones} \circ \underbrace{\Delta \circ H_*(S^1 | M)}_{\substack{\mathbb{S}^1\text{-ACTION} \\ \text{ON CONSTANT LOOPS}}} = 0$ ✓

Definition 4.2.6: Let \mathcal{C} be a symm. monoidal $(\infty, 2)$ -category. [THINK $\mathcal{C} = \text{Alg}(\text{COMP.}) = \left. \begin{array}{l} \text{ALG} \\ \text{BIMODULES} \\ \text{MORPH OF} \\ \text{BIMODULES} \\ \text{(CHAIN Cplx OF)} \end{array} \right\}$

- A CALABI-YAU OBJECT IN \mathcal{C} IS THE DATA OF
- (1) A EVALUABLE OBJECT $X \in \mathcal{C}$
 - (2) A 2-MORPHISM $\text{tr}: \text{ev}_X \circ \text{coev}_X \rightarrow 1$
- WHICH IS $\text{SO}(2)$ -EQUIVARIANT AND IS THE CUNIT FOR AN ADJUNCTION BETWEEN ev_X AND coev_X

Definition of $\text{Alg}(\text{Complexes})$: (AS $(\infty, 2)$ -CATEGORY)

Obj = dg ALGEBRAS A

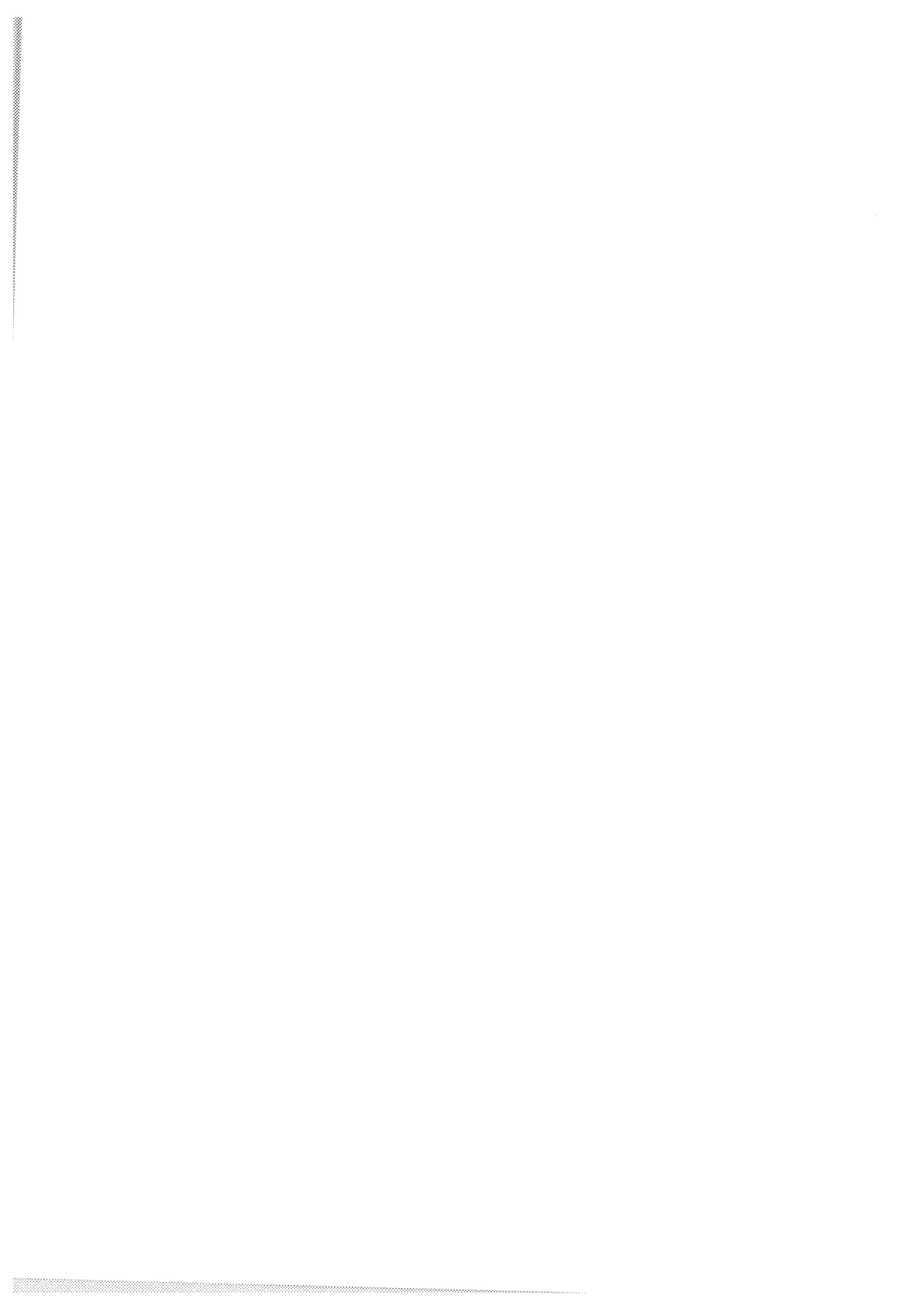
1-MORPH $(A, B) = \text{dg}(A, B)$ -BIMODULES

COMPOSITION: $A \xrightarrow{P} B \xrightarrow{Q} C \stackrel{\text{DEF}}{=} P \otimes_B Q$

2-MORPH $(A^P_B, A^{P'}_B) = \text{MORPH OF } (A, B)\text{-BIMODULES}$
 CHAIN Cplx OF

MONOIDAL STRUCTURE = TENSOR PRODUCT $A \otimes_{\mathbb{F}} A'$ OF ALG OF BIMODULES OF 2-MORPH

EVERY ALGEBRA A HAS A^{op} AS DUAL:



$$ev_A : A \otimes A^{op} \xrightarrow{A} \mathbb{F}$$

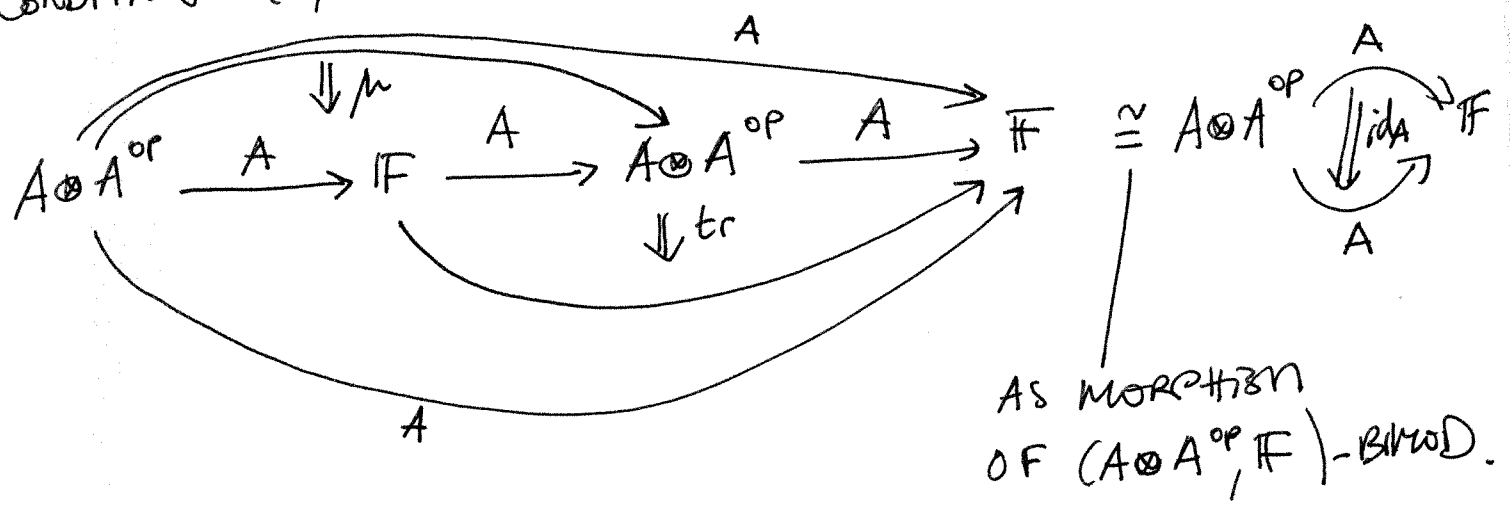
AS $(A \otimes A^{op}, \mathbb{F})$ -BIMODULE

$$coev_A : \mathbb{F} \xrightarrow{A} A \otimes A^{op}$$

AS $(\mathbb{F}, A \otimes A^{op})$ -BIMOD.

$$coev \xrightarrow{\quad} ev = ev_A \circ coev_A = \mathbb{F} \overset{R}{\otimes}_{A \otimes A^{op}} A \mathbb{F} = C_*(A, A)$$

CONDITION (2) OF CALABI-YAU :



\leadsto HYPOTHESIS OF BLUE THM

(THE OTHER WAY AROUND, DOESN'T KNOW HOW TO DEFINE μ .)