

# A BATALIN-VILKOVISKY ALGEBRA MORPHISM FROM DOUBLE LOOP SPACES TO FREE LOOPS

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ABSTRACT. Let  $M$  be a compact oriented  $d$ -dimensional smooth manifold and  $X$  a topological space. Chas and Sullivan [1] have defined a structure of Batalin-Vilkovisky algebra on  $\mathbb{H}_*(LM) := H_{*+d}(LM)$ . Getzler [8] has defined a structure of Batalin-Vilkovisky algebra on the homology of the pointed double loop space of  $X$ ,  $H_*(\Omega^2 X)$ . Let  $G$  be a topological monoid with a homotopy inverse. Suppose that  $G$  acts on  $M$ . We define a structure of Batalin-Vilkovisky algebra on  $H_*(\Omega^2 BG) \otimes \mathbb{H}_*(M)$  extending the Batalin-Vilkovisky algebra of Getzler on  $H_*(\Omega^2 BG)$ . We prove that the morphism of graded algebras

$$H_*(\Omega^2 BG) \otimes \mathbb{H}_*(M) \rightarrow \mathbb{H}_*(LM)$$

defined by Felix and Thomas [5], is in fact a morphism of Batalin-Vilkovisky algebras. In particular, if  $G = M$  is a connected compact Lie group, we compute the Batalin-Vilkovisky algebra  $\mathbb{H}_*(LG; \mathbb{Q})$ .

55P50, 55P35, 55P62

## 1. INTRODUCTION

We work over an arbitrary principal ideal domain  $\mathbb{k}$ .

Algebraic topology gives us two sources of Batalin-Vilkovisky algebras (Definition 6): Chas Sullivan string topology [1] and iterated loop spaces. More precisely, let  $X$  be a pointed topological space, extending the work of Cohen [3], Getzler [8] has shown that the homology  $H_*(\Omega^2 X)$  of the double pointed loop space on  $X$  is a Batalin-Vilkovisky algebra. Let  $M$  be a closed oriented manifold of dimension  $d$ . Denote by  $LM := \text{map}(S^1, M)$  the free loop space on  $M$ . By Chas and Sullivan [1], the (degree-shifted) homology  $\mathbb{H}_*(LM) := H_{*+d}(LM)$  of  $LM$  has also a Batalin-Vilkovisky algebra. In this paper, we related this two a priori very different Batalin-Vilkovisky algebras.

Initially, our work started with the following Theorem of Felix and Thomas.

**Theorem 1.** [5, Theorem 1] *Let  $G$  be a topological monoid acting on a smooth compact oriented manifold  $M$ . Consider the map  $\Theta_{G,M} : \Omega G \times M \rightarrow LM$  which sends the couple  $(w, m)$  to the free loop  $t \mapsto w(t).m$ .*

The map induced in homology

$$\mathbb{H}_*(\Theta_{G,M}) : H_*(\Omega G) \otimes \mathbb{H}_*(M) \rightarrow \mathbb{H}_*(LM)$$

is a morphism of commutative graded algebras.

In [5], Félix and Thomas stated this theorem with  $G = \text{aut}_1(M)$ , the monoid of self equivalences homotopic to the identity. But their theorem extends to any topological monoid  $G$  since any action on  $M$  factorizes through  $\text{Hom}(M, M)$  and  $\text{aut}_1(M)$  is the path component of the identity in  $\text{Hom}(M, M)$ . Note also that the assumptions of [5] “ $\mathbb{k}$  is a field ” and “ $M$  is simply connected” are not necessary.

If  $G = M$  is a Lie group,  $\Theta_{G,G} : \Omega G \times G \rightarrow LG$  is a homeomorphism. Therefore we recover the following well-known isomorphism (See for example [13, Proof of Theorem 10]).

**Corollary 2.** *Let  $G$  be a path-connected compact Lie group. The algebra  $\mathbb{H}_*(LG)$  is isomorphic to the tensor product of algebras  $H_*(\Omega G) \otimes \mathbb{H}_*(G)$ .*

The following is our main theorem.

**Theorem 3.** *(Theorem 15) Assume the hypothesis of Theorem 1. Suppose moreover that  $G$  has a homotopy inverse and that  $H_*(\Omega G)$  is torsion free. Then the tensor product of algebras  $H_*(\Omega G) \otimes \mathbb{H}_*(M)$  is a Batalin-Vilkovisky algebra and the morphism*

$$\mathbb{H}_*(\Theta_{G,M}) : H_*(\Omega G) \otimes \mathbb{H}_*(M) \rightarrow \mathbb{H}_*(LM)$$

defined by Felix and Thomas, is a morphism of Batalin-Vilkovisky algebras.

*Remark 4.* We assume that the homology with coefficients in  $\mathbb{k}$ ,  $H_*(\Omega G)$ , is torsion free (this hypothesis is of course satisfied if  $\mathbb{k}$  is a field), since we want  $H_*(\Omega G)$  to have a diagonal.

This Batalin-Vilkovisky algebra  $H_*(\Omega G) \otimes \mathbb{H}_*(M)$  contains  $H_*(\Omega G)$  as sub Batalin-Vilkovisky algebra (Corollary 23). Denote by  $BG$  the classifying space of  $G$ . We show that the sub Batalin-Vilkovisky algebra  $H_*(\Omega G)$  is isomorphic to the Batalin-Vilkovisky algebra  $H_*(\Omega^2 BG)$  introduced by Getzler (Proposition 14). So finally, we have obtained a morphism of Batalin-Vilkovisky (Theorem 24)

$$H_*(\Omega^2 BG) \rightarrow \mathbb{H}_*(LM)$$

from the homology of the double pointed loop space on  $BG$  to the free loop space homology on  $M$ .

Assuming that  $H_*(\Omega \text{aut}_1(M))$  is torsion free, Theorem 15 can also be applied to the monoid of self equivalences  $\text{aut}_1(M)$  as we shall now

explain. Since a smooth manifold has a CW-structure,  $M$  is a finite CW-complex. So by a theorem of Milnor [15],  $\text{Hom}(M, M)$  has the homotopy type of a CW-complex  $K$ . Therefore  $\text{aut}_1 M$  has the homotopy type of a path component of  $K$ . Recall that a path-connected homotopy associative  $H$ -space which has the homotopy of a CW-complex has naturally a homotopy inverse [20, Chapter X. Theorem 2.2], i.e. is naturally a  $H$ -group [17, p. 35]. Therefore  $\text{aut}_1 M$  has a  $H$ -group structure.

*Remark 5.* In [5], Felix and Thomas posed the following problem: Is

$$\mathbb{H}_*(\Theta_{\text{aut}_1(M), M}) : H_*(\Omega \text{aut}_1(M)) \otimes \mathbb{H}_*(M) \rightarrow \mathbb{H}_*(LM)$$

always surjective? The answer is no. Take  $M := S^{2n}$ ,  $n \geq 1$ . Rationally  $\text{aut}_1 S^{2n}$  has the same homotopy type as  $S^{4n-1}$ . Therefore  $H_*(\Omega \text{aut}_1 S^{2n}; \mathbb{Q}) \otimes H_*(S^{2n}; \mathbb{Q})$  is concentrated in even degree. It is easy to see that  $H_*(LS^{2n}; \mathbb{Q})$  is not trivial in odd degree. So  $H_*(\Theta_{\text{aut}_1(S^{2n}), S^{2n}})$  cannot be surjective.

We give now the plan of the paper:

**Section 2:** We give the definition and some examples of Batalin-Vilkovisky algebras.

**Section 3:** We compare the  $S^1$ -action on  $\Omega^2 BG$  to a  $S^1$ -action defined on  $\Omega G$ . Therefore the obvious isomorphism of algebras  $H_*(\Omega G) \rightarrow H_*(\Omega^2 BG)$  is in fact an isomorphism of Batalin-Vilkovisky algebras.

**Section 4:** The Batalin-Vilkovisky algebra  $H_*(\Omega G) \otimes \mathbb{H}_*(M)$  is introduced and related to  $\mathbb{H}_*(LM)$  (Theorem 15). Then extending the Theorem of Hepworth for Lie groups, we show how to compute its operator  $B$  (Proposition 18). This Batalin-Vilkovisky algebra structure on  $H_*(\Omega G) \otimes \mathbb{H}_*(M)$  is completely determined by its two sub Batalin-Vilkovisky algebras  $H_*(\Omega G)$  and  $\mathbb{H}_*(M)$  (Corollary 23 and Corollary 21) and by the bracket between them (Corollary 26).

**Section 5:** Extending the computations of the Batalin-Vilkovisky algebras  $\mathbb{H}_*(LS^1)$  and  $\mathbb{H}_*(LS^3)$ , we compute the Batalin-Vilkovisky algebras  $H_*(\Omega S^1) \otimes \mathbb{H}_*(M)$  and  $H_*(\Omega S^3) \otimes \mathbb{H}_*(M)$ .

**Section 6:** Using a second result of Felix and Thomas in [5], we show that  $\pi_{\geq 1}(\Omega \text{aut}_1(M)) \otimes \mathbb{Q}$  is always a sub Lie algebra of  $\mathbb{H}_*(LM; \mathbb{Q})$ .

**Section 7:** This section is devoted to calculations of the Batalin-Vilkovisky algebra  $H_*(\Omega G) \otimes \mathbb{H}_*(M)$  over the rationals. In particular, for any path-connected compact Lie group  $G$ , we compute  $\mathbb{H}_*(LG : \mathbb{Q})$ .

**Section 8:** This section is an appendix on split fibrations.

*Acknowledgment:* We would like to thank Yves Félix and Jean-Claude Thomas for helpful comments and for explaining us [5]. I would like also to thank Jim Stasheff, for sending me lists of corrections for

several papers (including this one). I would like especially to thank the referee for his serious job that led me to greatly improve this paper.

## 2. BATALIN-VILKOVISKY ALGEBRAS

**Definition 6.** A Batalin-Vilkovisky algebra is a commutative graded algebra  $A$  equipped with an action of the exterior algebra  $E(i_1) = H_*(S^1)$

$$H_*(S^1) \otimes A \rightarrow A$$

$$i_1 \otimes a \mapsto i_1 \cdot a \quad \text{denoted} \quad B(a)$$

such that

$$\begin{aligned} B(abc) &= B(ab)c + (-1)^{|a|}aB(bc) + (-1)^{(|a|-1)|b|}bB(ac) \\ &\quad - (Ba)bc - (-1)^{|a|}a(Bb)c - (-1)^{|a|+|b|}ab(Bc). \end{aligned}$$

The bracket  $\{ , \}$  of degree  $+1$  defined by

$$\{a, b\} = (-1)^{|a|} (B(ab) - (Ba)b - (-1)^{|a|}a(Bb))$$

for any  $a, b \in A$  satisfies the Poisson relation: For any  $a, b$  and  $c \in A$ ,

$$(7) \quad \{a, bc\} = \{a, b\}c + (-1)^{(|a|-1)|b|}b\{a, c\}.$$

Koszul [12, p. 3] (See also [8, Proposition 1.2] or [16]) has shown that  $\{ , \}$  is a Lie bracket and therefore that a Batalin-Vilkovisky algebra is a Gerstenhaber algebra.

*Example 8.* (Tensor product of Batalin-Vilkovisky algebras) Let  $A$  and  $A'$  be two Batalin-Vilkovisky algebras. Denote by  $B_A$  and  $B_{A'}$  their respective operators. Consider the tensor product of algebras  $A \otimes A'$ . Consider the operator  $B_{A \otimes A'}$  on  $A \otimes A'$  given by

$$B_{A \otimes A'}(x \otimes y) = B_A(x) \otimes y + (-1)^{|x|}x \otimes B_{A'}(y)$$

for  $x \in A$  and  $y \in A'$ . Then  $A \otimes A'$  equipped with  $B_{A \otimes A'}$  is a Batalin-Vilkovisky algebra.

Let  $X$  and  $X'$  be two pointed topological spaces. Let  $M$  and  $M'$  be two compact oriented smooth manifolds. It is easy to check that the Kunnet morphisms

$$H_*(\Omega^2 X) \otimes H_*(\Omega^2 X') \rightarrow H_*(\Omega^2(X \times X'))$$

$$\mathbb{H}_*(LM) \otimes \mathbb{H}_*(LM') \rightarrow \mathbb{H}_*(L(M \times M'))$$

are morphisms of Batalin-Vilkovisky algebras.

3. THE CIRCLE ACTION ON THE POINTED LOOPS OF A GROUP

In this section, we define an action up to homotopy of the circle  $S^1$  on the pointed loops  $\Omega G$  of a  $H$ -group  $G$ . When  $G$  is a monoid, we show (Propositions 13 and 14) that the algebra  $H_*(\Omega G)$  equipped with the operator induced by this action is a Batalin-Vilkovisky algebra isomorphic to the Batalin-Vilkovisky algebra  $H_*(\Omega^2 BG)$  introduced by Getzler [8]. Therefore, in this paper, instead of working with  $H_*(\Omega^2 BG)$ , we always consider  $H_*(\Omega G)$ .

Consider the free loop fibration  $\Omega X \xrightarrow{j} LX \xrightarrow{ev} X$ . The evaluation map  $ev : LX \rightarrow X$  admits a trivial section  $s : X \hookrightarrow LX$ .

Suppose now that  $X$  is a  $H$ -group  $G$ . The following is easy to check if  $X$  is a group. For the  $H$ -group case, see appendix 8. Consider the map  $r : LG \rightarrow \Omega G$  unique up to homotopy such that the composite

$$j \circ r : LG \xrightarrow{r} \Omega G \xrightarrow{j} LG$$

is homotopic to the map that sends the loop  $l \in LG$  to the loop  $t \in I \mapsto l(t)l(0)^{-1}$ . The map  $\Theta_{G,G} : \Omega G \times G \rightarrow LG$  that maps  $(w, g)$  to the free loop  $t \mapsto w(t)g$  is a homotopy equivalence. Its homotopy inverse is the map  $LG \rightarrow \Omega G \times G$ ,  $l \mapsto (r(l), l(0))$ . In particular,  $r : LG \rightarrow \Omega G$  is a retract of  $j$  and the composite  $r \circ s$  is homotopically trivial.

**Theorem 9.** *(Compare with [13, Proposition 28]) Let  $X$  be a topological space. The retract  $r : L\Omega X \rightarrow \Omega^2 X$  is a morphism of  $S^1$ -spaces up to homotopy (i.e. in the homotopy category of spaces).*

Theorem 9 follows from Propositions 10i), 12 and 13. In [13, Proposition 28]), the same theorem is proved but the retract was defined in a different way.

**Proposition 10.**

i) *Let  $G$  be a  $H$ -group. Then  $\Omega G$  is equipped with an action of  $S^1$  up to homotopy.*

ii) *Let  $G$  be a  $H$ -group acting up to homotopy on a topological space  $X$ . Then  $\Omega G \times X$  is equipped with an action of  $S^1$  up to homotopy such that  $\Theta_{G,X} : \Omega G \times X \rightarrow LX$  is a morphism of  $S^1$ -spaces up to homotopy.*

*Proof.* i) We define the action of  $S^1$  on  $\Omega G$  as the composition of

$$S^1 \times \Omega G \xrightarrow{S^1 \times j} S^1 \times LG \xrightarrow{action} LG \xrightarrow{r} \Omega G.$$

ii) We define the action of  $S^1$  on  $\Omega G \times X$  by  $s.(w, x) := (s.w, w(s).x)$ ,  $s \in S^1$ ,  $w \in \Omega$  and  $x \in X$ . Here  $s.w$  denote the action of  $s \in S^1$  on

$w \in \Omega G$  given in i). It is easy to see that  $\Theta_{G,X}$  is a morphism of  $S^1$ -spaces up to homotopy.  $\square$

**Proposition 11.** *Let  $G_1$  and  $G_2$  be two  $H$ -groups. Let  $f : (G_1, e) \rightarrow (G_2, e)$  be an homomorphism of  $H$ -spaces (in the sense of [17, p. 35]). Then  $\Omega f : \Omega(H_1, e) \rightarrow \Omega(H_2, e)$  is a morphism of  $S^1$ -spaces up to homotopy.*

*Proof.* A homomorphism of  $H$ -spaces between  $H$ -groups is necessarily a homomorphism of  $H$ -groups. The  $S^1$ -structure on  $\Omega G$  is clearly functorial with respect to homomorphisms of  $H$ -groups. Therefore the  $S^1$ -structure on  $\Omega G$  depends functorially only on the multiplication of  $G$ .  $\square$

**Proposition 12.** *The retract  $r : LG \rightarrow \Omega G$  is a morphism of  $S^1$ -spaces up to homotopy.*

*Proof.* Denote by  $p_{\Omega G} : \Omega G \times G \rightarrow \Omega G$  the projection on the first factor. By definition of  $r$ , the diagram

$$\begin{array}{ccc} LG & \xleftarrow[\simeq]{\Theta_{G,G}} & \Omega G \times G \\ & \searrow r & \swarrow p_{\Omega G} \\ & \Omega G & \end{array}$$

commutes up to homotopy. By Proposition 10 ii),  $\Theta_{G,G}$  is a morphism of  $S^1$ -spaces up to homotopy. By definition of the action of the circle  $S^1$  on  $\Omega G \times G$ , the projection  $p_{\Omega G}$  is also a morphism of  $S^1$ -spaces up to homotopy.  $\square$

**Proposition 13.** *Let  $(X, *)$  be a pointed space. The action of  $S^1$  up to homotopy on  $\Omega G$  when  $G = \Omega X$  given by Proposition 10 is homotopic to the action of  $S^1$  on  $\text{map}((E^2, S^1), (X, *))$  given by rotating the disk  $E^2$ .*

*Proof.* The map  $I \times S^1 \rightarrow E^2$  sending  $(t, x) \in \mathbb{R} \times \mathbb{C}$  to the barycenter of  $x$  with weight  $t$  and of 1 with weight  $1 - t$ , gives the canonical homeomorphism

$$\theta : \text{map}((E^2, S^1), (X, *)) \xrightarrow{\cong} \Omega \Omega X$$

defined by: for  $f \in \text{map}((E^2, S^1), (X, *))$ ,  $\theta(f)$  is the map sending  $x \in S^1 \subset \mathbb{C}$  to the loop  $t \in I \mapsto f(tx + 1 - t)$ .

Since  $j$  is a monomorphism in the homotopy category, to see that  $\theta$  commutes with the  $S^1$ -action up to homotopy, it suffices to see that the two maps  $j \circ \text{action} \circ (S^1 \times \theta)$  and  $j \circ \theta \circ \text{action}$  are homotopic.

The adjoint of  $j \circ \theta \circ action$  is the map

$$S^1 \times map((E^2, S^1), (X, *)) \times S^1 \times I \rightarrow X$$

$$(e^{is}, f, x, t) \mapsto f(e^{is}tx + (1-t)e^{is}).$$

By definition of  $r$ , the adjoint of  $j \circ action \circ (S^1 \times \theta)$  is the map

$$S^1 \times map((E^2, S^1), (X, *)) \times S^1 \times I \rightarrow X$$

$$(e^{is}, f, x, t) \mapsto \begin{cases} f((1-2t)e^{is} + 2t) & \text{if } t \leq \frac{1}{2}, \\ f((2t-1)e^{is}x + 2 - 2t) & \text{if } t \geq \frac{1}{2}. \end{cases}$$

The maps of pairs of spaces

$$(S^1 \times S^1 \times I, S^1 \times S^1 \times \{0, 1\}) \rightarrow (E^2, S^1)$$

$$(e^{is}, x, t) \rightarrow e^{is}tx + (1-t)e^{is}$$

and

$$(e^{is}, x, t) \mapsto \begin{cases} (1-2t)e^{is} + 2t & \text{if } t \leq \frac{1}{2}, \\ (2t-1)e^{is}x + 2 - 2t & \text{if } t \geq \frac{1}{2}. \end{cases}$$

are homotopic: to construct the homotopy, fill the triangle of vertices  $e^{is}$ ,  $e^{is}x$  and 1. Therefore  $j \circ action \circ (S^1 \times \theta)$  and  $j \circ \theta \circ action$  are homotopic.  $\square$

**Proposition 14.** *Let  $G$  be a topological monoid which is also a  $H$ -group. Then the algebra  $H_*(\Omega G)$  equipped with the  $H_*(S^1)$ -module structure given by Proposition 10i) is a Batalin-Vilkovisky algebra isomorphic to the Batalin-Vilkovisky algebra  $H_*(\Omega^2 BG)$  given by [8].*

*Proof.* Consider the classifying space of  $G$ ,  $BG$ . There exists a homomorphism of  $H$ -spaces  $h : G \xrightarrow{\cong} \Omega BG$  which is a weak homotopy equivalence since  $\pi_0(G)$  is a group. By Propositions 11 and 13,  $\Omega h : \Omega G \xrightarrow{\cong} \Omega^2 BG$  is a morphism of  $S^1$ -spaces up to homotopy. Therefore,  $H_*(\Omega h)$  is both an isomorphism of graded algebras and of  $H_*(S^1)$ -modules. Since  $H_*(\Omega^2 BG)$  is a Batalin-Vilkovisky algebra [8],  $H_*(\Omega G)$  is also a Batalin-Vilkovisky algebra.  $\square$

#### 4. THE BATALIN-VILKOVISKY ALGEBRA $H_*(\Omega^2 BG) \otimes \mathbb{H}_*(M)$

This section is the heart of the paper. We show (Theorem 15) that the tensor product of algebras  $H_*(\Omega G) \otimes \mathbb{H}_*(M)$  equipped with an operator  $B_{\Omega G \times M}$  is a Batalin-Vilkovisky algebra related to the Batalin-Vilkovisky algebra  $\mathbb{H}_*(LM)$  of Chas and Sullivan. Extending a result of Hepworth (Corollary 20), we give an explicit formula for this operator  $B_{\Omega G \times M}$ . We deduce then the morphism of Batalin-Vilkovisky algebra from  $H_*(\Omega G)$  to  $\mathbb{H}_*(LM)$  (Corollary 23).

Let us start by giving a short proof of the Felix and Thomas theorem.

*Proof of Theorem 1.* Since  $e.m = m$  for  $m \in M$ , the map  $\Theta_{G,M} : \Omega G \times M \rightarrow LM$  is a morphism of fiberwise monoids from the projection map  $p_M : \Omega G \times M \rightarrow M$  to the evaluation map  $ev : LM \rightarrow M$ . Therefore by [9, part 2) of Theorem 8.2], the composite

$$\mathbb{H}_*(\Theta_{G,M}) : H_*(\Omega G) \otimes \mathbb{H}_*(M) \rightarrow H_{*+d}(\Omega G \times M) \rightarrow \mathbb{H}_*(LM)$$

is a morphism of graded algebras.  $\square$

**Theorem 15.** *Let  $G$  be a topological monoid with a homotopy inverse, acting on a smooth compact oriented manifold  $M$ . Assume that  $H_*(\Omega G)$  is torsion free. Consider the operator  $B_{\Omega G \times M}$  given by the action of  $H_*(S^1)$  on  $H_*(\Omega G) \otimes \mathbb{H}_*(M)$  given by Proposition 10 ii). Then the tensor product of algebras  $H_*(\Omega G) \otimes \mathbb{H}_*(M)$  equipped with  $B_{\Omega G \times M}$  is a Batalin-Vilkovisky algebra such that*

$$H_*(\Theta_{G,M}) : H_*(\Omega G) \otimes \mathbb{H}_*(M) \rightarrow \mathbb{H}_*(LM)$$

*is a morphism of Batalin-Vilkovisky algebras.*

*Remark 16.* Suppose that  $G = M$  is a Lie group. Then, with Corollary 2, since  $\Theta_{G,G}$  is an  $S^1$ -equivariant homeomorphism, Theorem 15 is obvious.

*Proof.* Since  $H_*(\Omega G)$  is assumed to be torsion free, the Kunneth morphism is an isomorphism. Therefore the algebra  $H_*(\Omega G) \otimes \mathbb{H}_*(LM)$  can be identified with the algebra  $H_{*+d}(\Omega G \times LM)$ . For  $w \in \Omega G$  and  $l \in LM$ , denote by  $w * l$  the free loop on  $M$  defined by  $w * l(t) = w(t).l(t)$ , for  $t \in S^1$ . Let  $\Phi_{G,M} : \Omega G \times LM \rightarrow \Omega G \times LM$  be the map sending  $(w, l)$  to  $(w, w * l)$ . Since  $G$  has an homotopy inverse,  $\Phi_{G,M}$  is a homotopy equivalence. Since  $\Phi_{G,M}$  is a morphism of fiberwise monoids, by [9, part 2) of Theorem 8.2],

$$H_*(\Phi_{G,M}) : H_*(\Omega G) \otimes \mathbb{H}_*(LM) \rightarrow H_*(\Omega G) \otimes \mathbb{H}_*(LM)$$

is an isomorphism of algebras.

Consider the action of  $S^1$  on  $\Omega G$  up to homotopy given by Proposition 10 i) and the action of  $S^1$  on  $LM$  given by rotation of the loops. Consider the induced *diagonal* action of  $S^1$  on  $\Omega G \times LM$ . Explicitly, if  $G$  is a group, the diagonal action of  $s \in S^1$  on  $(w, l) \in \Omega G \times LM$  is simply given by the pointed loop  $t \mapsto w(t+s)w(s)^{-1}$  and the free loop  $t \mapsto l(t+s)$ .

Consider the *twisted* action of  $S^1$  on  $\Omega G \times LM$  defined by  $s.(w, l) = (s.w, t \mapsto w(s).l(t+s))$  where  $s.w$  is the action of  $s \in S^1$  on  $w \in \Omega G$  given by Proposition 10 i). With respect to the twisted action on the source and the diagonal action on the target,  $\Phi_{G,M}$  is a morphism of  $S^1$ -spaces up to homotopy.



The algebra  $H_*(\Omega G) \otimes \mathbb{H}_*(LM)$  equipped with the  $H_*(S^1)$ -module structure given by the diagonal action is the tensor product of the Batalin-Vilkovisky algebra  $H_*(\Omega G)$  given by Proposition 14 and of the Batalin-Vilkovisky algebra  $\mathbb{H}_*(LM)$  given by Chas and Sullivan [1]. Therefore by Example 8, it is a Batalin-Vilkovisky algebra.

Since the isomorphism  $H_*(\Phi_{G,m})$  is both a morphism of algebras and a morphism of  $H_*(S^1)$ -modules,  $H_*(\Omega G) \otimes \mathbb{H}_*(LM)$  equipped with the  $H_*(S^1)$ -module structure given by the twisted action is also a Batalin-Vilkovisky algebra.

Consider the trivial section  $s : M \hookrightarrow LM$  mapping  $x \in M$  to the free loop constant on  $x$ . It is well-known that  $\mathbb{H}_*(s) : \mathbb{H}_*(M) \rightarrow \mathbb{H}_*(LM)$  is a morphism of algebras. The map  $\Omega G \times s : \Omega G \times M \rightarrow \Omega G \times LM$  is  $S^1$ -equivariant with respect to the  $S^1$ -action on  $\Omega G \times M$  given by Proposition 10 ii) and to the twisted  $S^1$ -action on  $\Omega G \times LM$ . Therefore, since  $H_*(\Omega G) \otimes \mathbb{H}_*(s)$  is injective,  $H_*(\Omega G) \otimes \mathbb{H}_*(M)$  is a Batalin-Vilkovisky algebra and  $H_*(\Omega G) \otimes \mathbb{H}_*(s)$  is a morphism of Batalin-Vilkovisky algebras.

The composite

$$\Omega G \times M \xrightarrow{\Omega G \times s} \Omega G \times LM \xrightarrow{\Phi_{G,M}} \Omega G \times LM \xrightarrow{proj} LM$$

is  $\Theta_{G,M}$ . Therefore  $H_*(\Theta_{G,M})$  is the composite of the following morphisms of Batalin-Vilkovisky algebras:

$$\begin{aligned} H_*(\Omega G) \otimes \mathbb{H}_*(M) &\xrightarrow{H_*(\Omega G) \otimes \mathbb{H}_*(s)} H_*(\Omega G) \otimes \mathbb{H}_*(LM) \\ &\xrightarrow{\mathbb{H}_*(\Phi_{G,M})} H_*(\Omega G) \otimes \mathbb{H}_*(LM) \xrightarrow{\varepsilon_{H_*(\Omega G)} \otimes \mathbb{H}_*(LM)} \mathbb{H}_*(LM) \end{aligned}$$

Note that  $\varepsilon_{H_*(\Omega G)} \otimes \mathbb{H}_*(LM)$  is a morphism of Batalin-Vilkovisky algebras since the augmentation  $\varepsilon_{H_*(\Omega G)} : H_*(\Omega G) \rightarrow \mathbb{k}$  is a morphism of Batalin-Vilkovisky algebras. So finally,  $H_*(\Theta_{G,M})$  is a morphism of Batalin-Vilkovisky algebras.  $\square$

**Definition 17.** [10] Let  $X$  be a pointed space. Let  $\sigma : S^1 \times \Omega X \rightarrow X$ ,  $(s, w) \mapsto w(s)$ , denote the evaluation map. The *homology suspension* is the morphism of degree +1,  $\sigma_* : H_q(\Omega X) \rightarrow H_{q+1}(X)$  defined by  $\sigma_*(a) = H_*(\sigma)([S^1] \otimes a)$ ,  $a \in H_q(\Omega X)$ ,  $q \geq 0$ .

According to [10, Lemma 7], this homology suspension coincides with the usual one studied in [20, Chapter VIII]. Since this paper was written almost completely before the preprint of Hepworth appeared, we will never use in this paper this fact that regrettably, we did not notice. However, we felt that it was necessary to use his terminology and we rewrote our paper accordingly.

In [10, Theorem 5], Hepworth computed the Batalin-Vilkovisky algebra on the modulo 2 free loop space homology on the special orthogonal group,  $\mathbb{H}_*(LSO(n); \mathbb{F}_2)$ . When  $n \geq 4$ , Lemma 7 of [10] is required in order to achieve this interesting computation.

**Proposition 18.** *Let  $G$  be a  $H$ -group. Let  $X$  be a topological space. Let  $act_X : G \times X \rightarrow X$  be an action up to homotopy of  $G$  on  $X$ . Suppose that  $H_*(\Omega G)$  is torsion free. Denote by  $B_{\Omega G}$  (respectively  $B_{\Omega G \times X}$ ) the operator given by the action of  $H_*(S^1)$  on  $H_*(\Omega G)$  (respectively  $H_*(\Omega G \times X)$ ) given by Proposition 10. Then for any  $a \in H_*(\Omega G)$ ,  $x \in H_*(X)$ ,*

$$B_{\Omega G \times X}(a \otimes x) = (B_{\Omega G}a) \otimes x + \sum (-1)^{|a_{(1)}|} a_{(1)} \otimes (\sigma_*(a_{(2)}) \cdot x).$$

Here  $\Delta a = \sum a_{(1)} \otimes a_{(2)}$  is the diagonal of  $a \in H_*(\Omega G)$ .

*Proof.* By Proposition 10 ii), the action of  $H_*(S^1)$  on  $H_*(\Omega G) \otimes H_*(X)$  is the composite

$$\begin{array}{c} H_*(S^1) \otimes H_*(\Omega G) \otimes H_*(X) \\ \downarrow \Delta_{H_*(S^1) \otimes H_*(\Omega G) \otimes H_*(X)} \\ H_*(S^1) \otimes H_*(\Omega G) \otimes H_*(S^1) \otimes H_*(\Omega G) \otimes H_*(X) \\ \downarrow act_{H_*(\Omega G) \otimes H_*(\sigma) \otimes H_*(X)} \\ H_*(\Omega G) \otimes H_*(G) \otimes H_*(X) \\ \downarrow H_*(\Omega G) \otimes act_{H_*(X)} \\ H_*(\Omega G) \otimes H_*(X) \end{array}$$

where  $\Delta_{H_*(S^1) \otimes H_*(\Omega G)}$  is the diagonal of  $H_*(S^1) \otimes H_*(\Omega G)$  and  $act_{H_*(\Omega G)} : H_*(S^1) \otimes H_*(\Omega G) \rightarrow H_*(\Omega G)$  is the action of  $H_*(S^1)$  on  $H_*(\Omega G)$  given by Proposition 10 i).

Since

$$\begin{aligned} \Delta_{H_*(S^1) \otimes H_*(\Omega G)}([S^1] \otimes a) = \\ \sum [S^1] \otimes a_{(1)} \otimes 1 \otimes a_{(2)} + \sum (-1)^{|a_{(1)}|} 1 \otimes a_{(1)} \otimes [S^1] \otimes a_{(2)}, \end{aligned}$$

$$\begin{aligned} B_{\Omega G \times X}(a \otimes x) = \sum (B_{\Omega G}a_{(1)}) \otimes H_*(\sigma)(1 \otimes a_{(2)}) \cdot x \\ + \sum (-1)^{|a_{(1)}|} a_{(1)} \otimes H_*(\sigma)([S^1] \otimes a_{(2)}) \cdot x. \end{aligned}$$

Let  $\varepsilon_{H_*(\Omega G)} : H_*(\Omega G) \rightarrow \mathbb{k}$  be the augmentation of the Hopf algebra  $H_*(\Omega G)$ . Since the restriction of  $\sigma$  to  $\{0\} \times \Omega G$  is the composite  $\Omega G \rightarrow$

$\{e\} \rightarrow G$ ,

$$H_*(\sigma)(1 \otimes a) = \varepsilon_{H_*(\Omega G)}(a)1.$$

In a Hopf algebra,  $(Id \otimes \varepsilon) \circ \Delta = Id$ . Therefore,

$$\sum (B_{\Omega G}a_{(1)}) \otimes H_*(\sigma)(1 \otimes a_{(2)}) \cdot x = \sum (B_{\Omega G}a_{(1)}) \otimes \varepsilon_{H_*(\Omega G)}(a_{(2)})x = (B_{\Omega G}a) \otimes x.$$

On the other hand, by Definition 17

$$\sum (-1)^{|a_{(1)}|} a_{(1)} \otimes H_*(\sigma)([S^1] \otimes a_{(2)}) \cdot x = \sum (-1)^{|a_{(1)}|} a_{(1)} \otimes (\sigma_*(a_{(2)}) \cdot x).$$

□

**Lemma 19.** *Let  $G$  be a path-connected Lie group. Then  $H_*(\Omega G)$  is  $\mathbb{k}$ -free and concentrated in even degree. So for any  $a \in H_*(\Omega G)$ ,  $B_{\Omega G}a = 0$ .*

*Proof.* Let  $\Omega_0 G$  be the path-component of the constant loop  $\hat{e}$  in  $\Omega G$ . Since  $(\Omega G, \hat{e})$  is a  $H$ -group, the composite of the inclusion map and of the multiplication

$$\mathbb{k}[\pi_0(\Omega G)] \otimes H_*(\Omega_0 G) \hookrightarrow H_0(\Omega G) \otimes H_*(\Omega G) \rightarrow H_*(\Omega G)$$

is an isomorphism (of algebras since  $H_*(\Omega G)$  is commutative). Let  $\tilde{G}$  be the universal cover of  $G$ . Then we have an isomorphism of algebras  $H_*(\Omega \tilde{G}) \xrightarrow{\cong} H_*(\Omega_0 \tilde{G})$ . Since  $\tilde{G}$  is a simply-connected Lie group, by a result of Bott [14, Theorem 21.7 and Remark 2],  $H_*(\Omega \tilde{G})$  is  $\mathbb{k}$ -free and concentrated in even degree. Therefore  $H_*(\Omega G) \cong \mathbb{k}[\pi_1(G)] \otimes H_*(\Omega \tilde{G})$  is also  $\mathbb{k}$ -free and concentrated in even degree. Therefore  $B_{\Omega G} : H_*(\Omega G) \rightarrow H_{*+1}(\Omega G)$  is trivial. □

From Remark 16, Proposition 18 and Lemma 19, since here  $(-1)^{|a_{(1)}|}$  is equal to 1, we immediately obtain the following Corollary due to Hepworth.

**Corollary 20.** [10, Theorem 1] *Let  $G$  be a path-connected compact Lie group. Then as Batalin-Vilkovisky algebra,*

$$\mathbb{H}_*(LG) \cong H_*(\Omega G) \otimes \mathbb{H}_*(G)$$

and  $B(a \otimes x) = \sum a_{(1)} \otimes (\sigma_*(a_{(2)}) \cdot x)$  for  $a \in H_*(\Omega G)$ ,  $x \in \mathbb{H}_*(G)$ .

**Corollary 21.** *Let  $G$  be a  $H$ -group acting up to homotopy on a topological space  $X$ . Assume that  $H_*(\Omega G)$  is torsion free. Then  $H_*(X)$  is a trivial sub  $H_*(S^1)$ -module of the  $H_*(S^1)$ -module  $H_*(\Omega G) \otimes H_*(X)$  given by Proposition 10 ii).*

Note that the composite  $X \cong \{\hat{e}\} \times X \rightarrow \Omega G \times X \xrightarrow{\Theta_{G,X}} LX$  is homotopic to  $s : X \hookrightarrow LX$  the trivial section mapping  $x \in X$  to  $\hat{x}$  the free loop constant on  $x$ . Through  $s$ ,  $X$  is a trivial sub  $S^1$ -space of  $LX$ .

*First proof of Corollary 21 without using Proposition 18.* By definition, in Proposition 10 i), the action of any  $t \in S^1$  on the constant loop  $\hat{e} \in \Omega(G, e)$  is  $\hat{e}$ , since  $r : LG \rightarrow \Omega G$  is a pointed map. Therefore by definition, in Proposition 10 ii), the action of  $t \in S^1$  on  $(\hat{e}, x) \in \Omega G \times X$  is  $t.(\hat{e}, x) = (t.\hat{e}, \hat{e}(s).x) = (\hat{e}, e.x)$ . Therefore  $X \cong \{\hat{e}\} \times X$  is up to homotopy a trivial sub  $S^1$ -space of  $\Omega G \times X$ . And in homology, if  $H_*(X)$  is considered as a trivial  $H_*(S^1)$ -module, the morphism  $H_*(X) \rightarrow H_*(\Omega G) \otimes H_*(X)$ ,  $x \mapsto 1 \otimes x$ , is a morphism of  $H_*(S^1)$ -modules.  $\square$

*Second proof of Corollary 21 using Proposition 18.* Again, we observe that the constant loop  $\hat{e} \in \Omega G$  on the neutral element is a fixed point under the  $S^1$ -action of  $\Omega G$ . Therefore  $B_{\Omega G}(1) = 0$ .

Let  $\varepsilon_{H_*(S^1)} : H_*(S^1) \rightarrow \mathbb{k}$  be the augmentation. Since the restriction of  $\sigma$  to  $S^1 \times \{\hat{e}\}$  is the composite  $S^1 \rightarrow \{e\} \rightarrow G$ ,

$$(22) \quad \sigma_*(1) := H_*(\sigma)([S^1] \otimes 1) = \varepsilon_{H_*(S^1)}([S^1])1 = 0.$$

By Proposition 18,

$$B_{\Omega G \times X}(1 \otimes x) = (B_{\Omega G}1) \otimes x + 1 \otimes (\sigma_*(1) \cdot x) = 0 + 0.$$

$\square$

**Corollary 23.** *Let  $G$  be a  $H$ -group acting up to homotopy on a smooth compact oriented manifold  $M$ . Assume that  $H_*(\Omega G)$  is torsion free. Then  $H_*(\Omega G)$  is a sub  $H_*(S^1)$ -module of the  $H_*(S^1)$ -module  $H_*(\Omega G) \otimes \mathbb{H}_*(M)$  given by Proposition 10 ii).*

*First proof.* For any  $a \in H_*(\Omega G)$ , since  $\sigma_*(a_{(2)})$  has positive degree, its action on the fundamental class  $[M]$  is null. Therefore  $\sum (-1)^{|a_{(1)}|} a_{(1)} \otimes \sigma_*(a_{(2)}) \cdot [M] = 0$ . So by Proposition 18,  $B_{\Omega G \times M}(a \otimes [M]) = (B_{\Omega G}a) \otimes [M] + 0$ . Therefore, we have proved

$$Id \otimes [M] : H_*(\Omega G) \rightarrow H_*(\Omega G) \otimes \mathbb{H}_*(M)$$

is a morphism of  $H_*(S^1)$ -modules.  $\square$

*Second proof using shriek maps.* The trivial fibration  $proj : \Omega G \times M \rightarrow \Omega G$  is  $S^1$ -equivariant. Therefore by [2, Section 2.3 Borel Construction], the integration along the fiber of  $proj$ ,  $proj_! : H_*(\Omega G) \rightarrow H_*(\Omega G) \otimes \mathbb{H}_*(LM)$  is  $H_*(S^1)$ -linear. But  $proj_! = Id \otimes [M]$ .  $\square$

**Theorem 24.** *Assume the hypothesis of Theorem 15. Then the composite*

$$H_*(\Omega^2 BG) \cong H_*(\Omega G) \xrightarrow{Id \otimes [M]} H_*(\Omega G) \otimes \mathbb{H}_*(M) \xrightarrow{\mathbb{H}_*(\Theta_{G,M})} \mathbb{H}_*(LM)$$

*is a morphism of Batalin-Vilkovisky algebras.*

As pointed in the introduction of this paper, this theorem can be deduce using Theorem 15. But we prefer to give an independent proof.

*Proof without using Theorem 15.* By Proposition 14, the obvious isomorphism of algebras between  $H_*(\Omega^2 BG)$  and  $H_*(\Omega G)$  is an isomorphism of Batalin-Vilkovisky algebra. By Corollary 23, the inclusion of algebras

$$Id \otimes [M] : H_*(\Omega G) \rightarrow H_*(\Omega G) \otimes \mathbb{H}_*(M)$$

is also a morphism of  $H_*(S^1)$ -modules. By Theorem 1,  $\mathbb{H}_*(\Theta_{G,M})$  is a morphism of algebras. By Proposition 10 ii),  $\mathbb{H}_*(\Theta_{G,M})$  is also a morphism of  $H_*(S^1)$ -modules.  $\square$

From Theorem 24 and Lemma 19, we obtain:

**Corollary 25.** *Let  $G$  be a path-connected compact Lie group. Then the algebra  $H_*(\Omega^2 BG)$  equipped with the operator  $B = 0$  is a sub Batalin-Vilkovisky algebra of  $\mathbb{H}_*(LG)$ .*

**Corollary 26.** *Assume the hypothesis of Theorem 15. Let  $a \in H_*(\Omega G)$  and  $x \in \mathbb{H}_*(M)$ . Then the Lie bracket of  $a \otimes [M]$  and  $1 \otimes x$  in the Batalin-Vilkovisky algebra  $H_*(\Omega G) \otimes \mathbb{H}_*(M)$  is given by*

$$\{a \otimes [M], 1 \otimes x\} = \sum (-1)^{|a_{(2)}|} a_{(1)} \otimes \sigma_*(a_{(2)}) \cdot x.$$

*In particular, if  $a$  is primitive,  $\{a \otimes [M], 1 \otimes x\} = (-1)^{|a|} 1 \otimes \sigma_*(a) \cdot x$ .*

*Proof.* Since by Corollary 23 and Corollary 21,  $H_*(\Omega G)$  and  $\mathbb{H}_*(M)$  are two sub-Batalin-Vilkovisky algebras of the Batalin-Vilkovisky algebra  $H_*(\Omega G) \otimes \mathbb{H}_*(M)$ ,

$$(27) \quad \{a \otimes [M], 1 \otimes x\} = (-1)^{|a|} B_{\Omega G \times M}(a \otimes x) - (-1)^{|a|} (B_{\Omega G} a) \otimes x - a \times 0.$$

Therefore, by Proposition 18,

$$\{a \otimes [M], 1 \otimes x\} = \sum (-1)^{|a_{(2)}|} a_{(1)} \otimes \sigma_*(a_{(2)}) \cdot x.$$

In particular if  $\Delta a = a \otimes 1 + 1 \otimes a$ ,

$$\{a \otimes [M], 1 \otimes x\} = a \otimes \sigma_*(1) \cdot x + (-1)^{|a|} 1 \otimes \sigma_*(a) \cdot x.$$

By equation (22),  $\sigma_*(1) = 0$ . Therefore  $\{a \otimes [M], 1 \otimes x\} = 0 + (-1)^{|a|} 1 \otimes \sigma_*(a) \cdot x$ .  $\square$

## 5. SOME COMPUTATIONS

Using Hepworth's definition of the homology suspension  $\sigma_*$  (Definition 17), Lemma 11 of [13] becomes the well-known fact.

**Lemma 28.** *Let  $X$  be a pointed topological space. Let  $n \geq 0$ . Denote by  $hur_X : \pi_n(X) \rightarrow H_n(X)$  the Hurewicz map. We have the commutative diagram*

$$\begin{array}{ccc} \pi_n(\Omega X) & \xrightarrow{\cong} & \pi_{n+1}(X) \\ hur_{\Omega X} \downarrow & & \downarrow hur_X \\ H_n(\Omega X) & \xrightarrow{\sigma_*} & H_{n+1}(X) \end{array}$$

where  $\pi_n(\Omega X) \cong \pi_{n+1}(X)$  is the adjunction map.

In [13, Theorem 10] and then in [10, Proposition 9], this Lemma was used to compute the Batalin-Vilkovisky algebra  $\mathbb{H}_*(LS^1)$ . The same proof shows the following Proposition.

**Proposition 29.** *Let  $M$  be a compact oriented manifold equipped with an action of the circle  $S^1$ . Denote by  $x$  a generator of  $\mathbb{Z}$ . Then the Batalin-Vilkovisky algebra  $H_*(\Omega S^1) \otimes \mathbb{H}_*(M)$  is the tensor product of graded algebras  $\mathbb{k}[\mathbb{Z}] \otimes \mathbb{H}_*(M)$  with  $B_{\Omega S^1 \times M}(x^i \otimes m) = ix^i \otimes ([S^1].m)$  for any  $i \in \mathbb{Z}$ ,  $m \in \mathbb{H}_*(M)$ .*

*Proof.* By applying Lemma 28, to the degree  $i$  map  $S^1 \rightarrow S^1$ , we obtain that  $\sigma_*(x^i) = i[S^1]$ . By applying Proposition 18 and Lemma 19, we conclude.  $\square$

The following Proposition generalises the computation of the Batalin-Vilkovisky algebra  $\mathbb{H}_*(LS^3)$  due independently to the author [13, Theorem 16] and to Tamanoi [18].

**Proposition 30.** *Let  $M$  be a compact oriented manifold equipped with an action of the sphere  $S^3$ . Then the Batalin-Vilkovisky algebra  $H_*(\Omega S^3) \otimes \mathbb{H}_*(M)$  is the tensor product of graded algebras  $\mathbb{k}[u_2] \otimes \mathbb{H}_*(M)$  with  $B_{\Omega S^3 \times M}(u_2^i \otimes m) = iu_2^{i-1} \otimes ([S^3].m)$  for any  $i \in \mathbb{N}$ ,  $m \in \mathbb{H}_*(M)$ .*

*Proof.* Let  $ad : S^2 \rightarrow \Omega S^3$  be the adjoint map and let  $u_2$  be  $H_*(ad)([S^2])$ , the image of  $[S^2]$  by  $H_*(ad)$ . By Bott-Samelson theorem,  $H_*(\Omega S^3)$  is the polynomial algebra  $\mathbb{k}[u_2]$ . By Lemma 28, we obtain that  $\sigma_*(u_2) = [S^3]$ . Therefore since  $u_2$  is primitive, by Corollary 26,  $\{u_2 \otimes [M], 1 \otimes m\} = 1 \otimes ([S^3].m)$ . By Corollary 21,  $B_{\Omega S^3 \times M}(1 \otimes m) = 0$ . By Corollary 23,  $B_{\Omega S^3 \times M}(u_2^i \otimes [M]) = B_{\Omega S^3}(u_2^i) \otimes [M] = 0$ . Using the Poisson relation (7),  $\{u_2^i \otimes [M], 1 \otimes m\} = iu_2^{i-1} \otimes ([S^3].m)$ . Therefore  $B(u_2^i \otimes m) = iu_2^{i-1} \otimes ([S^3].m)$ .  $\square$

In [10, Proposition 10], Hepworth gave a new proof for the computation of the Batalin-Vilkovisky algebra  $\mathbb{H}_*(LS^3)$ . This proof of Hepworth gives also a proof of Proposition 30.

### 6. A SUB LIE ALGEBRA OF $\mathbb{H}_*(LM; \mathbb{Q})$

Let  $M$  be a smooth compact oriented manifold. We show that the rational free loop homology on  $M$ ,  $\mathbb{H}_*(LM; \mathbb{Q})$ , equipped with the loop bracket contains a sub Lie algebra.

**Definition 31.** ([7, p. 272], [19, 7.4.9], [11, p. 235]) Let  $L$  be a graded Lie algebra. Let  $N$  be a left  $L$ -module. Then the direct product of graded  $\mathbb{k}$ -modules  $N \times L$  can be equipped with a Lie bracket defined by

$$\{(m, l), (m', l')\} := (l.m' - (-1)^{|l'||m|}l'.m, \{l, l'\})$$

for  $m, m' \in N$  and  $l, l' \in L$ . This graded Lie algebra is denoted  $N \rtimes L$  and is called the *semidirect product* of  $N$  and  $L$  or *trivial extension* of  $L$  by  $N$ .

*Example 32.* (The Lie bracket of degree +1 on  $s^{-1}\pi_{\geq 2}(G) \otimes \mathbb{k} \oplus \mathbb{H}_*(M)$ ) Let  $G$  be a path-connected topological monoid acting on a smooth compact oriented manifold  $M$ . The associative algebra  $H_*(G)$  acts on  $\mathbb{H}_*(M)$  and therefore on  $s\mathbb{H}_*(M)$  by  $a.sx := (-1)^{|a|-1}s(ax)$  for  $a \in H_*(G)$  and  $x \in \mathbb{H}_*(M)$ . Since  $G$  is a path-connected topological monoid, by [20, X.6.3], the Hurewicz morphism  $hur_G : \pi_*(G) \rightarrow H_*(G)$  is a morphism of graded Lie algebras from the Samelson product to the commutator associated to the associative algebra  $H_*(G)$ . The truncated homotopy groups of  $G$ ,  $\pi_{\geq 2}(G)$  is a sub-Lie algebra of  $\pi_*(G)$ . So finally, the composite

$$\pi_{\geq 2}(G) \hookrightarrow \pi_*(G) \xrightarrow{hur_G} H_*(G) \rightarrow \text{Hom}(s\mathbb{H}_*(M), s\mathbb{H}_*(M))$$

is a morphism of graded Lie algebras, i.e.  $s\mathbb{H}_*(M)$  is a module over the Lie algebra  $\pi_{\geq 2}(G)$ . Therefore by the definition of the semidirect product  $s\mathbb{H}_*(M) \rtimes \pi_{\geq 2}(G)$  (Definition 31 above),  $s\mathbb{H}_*(M) \oplus \pi_{\geq 2}(G)$  is a graded Lie algebra. By desuspending,  $\mathbb{H}_*(M) \oplus s^{-1}\pi_{\geq 2}(G)$  has a Lie bracket of degree +1.

Explicitly the bracket on  $s^{-1}\pi_{\geq 2}(G) \otimes \mathbb{k} \oplus \mathbb{H}_*(M)$  is defined by

- $\{s^{-1}f, s^{-1}g\} := s^{-1}\{f, g\}$ , (here  $\{f, g\}$  is the Samelson bracket of  $f$  and  $g \in \pi_{\geq 2}(G) \otimes \mathbb{k}$ )
- $\{x, y\} := 0$  for  $x$  and  $y \in \mathbb{H}_*(M)$ ,
- $\{s^{-1}f, x\} := (-1)^{|f|-1}hur_G f.x$ . (Here  $hur_G : \pi_*(G) \otimes \mathbb{k} \rightarrow H_*(G)$  is the Hurewicz morphism.)

**Theorem 33.** *Let  $G$  be a path-connected topological monoid with an homotopy inverse acting on a smooth compact oriented manifold  $M$ . Assume that  $H_*(\Omega G)$  is torsion free. Let  $\tilde{\Gamma}_1 : s^{-1}\pi_{\geq 2}(G) \otimes \mathbb{k} \rightarrow \mathbb{H}_*(LM)$  be the composite of the adjunction map, the Hurewicz morphism and of the map considered in Theorem 24*

$$\pi_{n+1}G \otimes \mathbb{k} \cong \pi_n \Omega G \otimes \mathbb{k} \xrightarrow{hur_{\Omega G}} H_n(\Omega G) \xrightarrow{Id \otimes [M]} H_n(\Omega G) \otimes \mathbb{H}_0(M) \xrightarrow{\mathbb{H}_n(\Theta_{G,M})} \mathbb{H}_n(LM)$$

for  $n \geq 1$ .

Then the  $\mathbb{k}$ -linear morphism  $s^{-1}\pi_{\geq 2}(G) \otimes \mathbb{k} \oplus \mathbb{H}_*(M) \rightarrow \mathbb{H}_*(LM)$  mapping  $(s^{-1}f, x)$  to  $\tilde{\Gamma}_1(s^{-1}f) + \mathbb{H}_*(s)(x)$  is a morphism of Lie algebras between the Lie bracket from example 32 and the loop bracket of  $LM$ .

Recall that  $s : M \hookrightarrow LM$  denotes the trivial section.

*Proof.* Recall from the proof of Proposition 14, that there exists a homomorphism of  $H$ -spaces  $h : G \xrightarrow{\cong} \Omega BG$  which is also a weak equivalence and that  $H_*(\Omega h) : H_*(\Omega G) \xrightarrow{\cong} H_*(\Omega^2 BG)$  is an isomorphism of Batalin-Vilkovisky algebras. Since  $h : G \xrightarrow{\cong} \Omega BG$  is a homomorphism of  $H$ -spaces between two path-connected homotopy associative  $H$ -spaces,  $\pi_*(h) : \pi_*(G) \xrightarrow{\cong} \pi_*(\Omega BG)$  is a morphism of graded Lie algebras with respect to the Samelson brackets on  $G$  and on  $\Omega BG$ . For  $n \geq 1$ , consider the commutative diagram of graded  $\mathbb{k}$ -modules

$$\begin{array}{ccccc} \pi_{n+1}(\Omega BG) \otimes \mathbb{k} & \xrightarrow{\cong} & \pi_n(\Omega^2 BG) \otimes \mathbb{k} & \xrightarrow{hur_{\Omega^2 BG}} & H_n(\Omega^2 BG) \\ \pi_{n+1}(h) \otimes \mathbb{k} \uparrow & & \pi_n(\Omega h) \otimes \mathbb{k} \uparrow & & \uparrow H_n(\Omega h) \\ \pi_{n+1}(G) \otimes \mathbb{k} & \xrightarrow{\cong} & \pi_n(\Omega G) \otimes \mathbb{k} & \xrightarrow{hur_{\Omega G}} & H_n(\Omega G) \end{array}$$

By [3, Remark 1.2 p. 214-5], the top line is a morphism of Lie algebras between the Samelson bracket and the Browder bracket. Therefore, the bottom line is also a morphism of Lie algebras. By Theorem 24, the composite

$$H_n(\Omega G) \xrightarrow{Id \otimes [M]} H_n(\Omega G) \otimes \mathbb{H}_0(M) \xrightarrow{\mathbb{H}_n(\Theta_{G,M})} \mathbb{H}_n(LM)$$

is a morphism of Lie algebras. Therefore, by composition,

$$\tilde{\Gamma}_1 : s^{-1}\pi_{n+1}(G) \otimes \mathbb{k} \rightarrow \mathbb{H}_n(LM)$$

is a morphism of Lie algebras between the Samelson bracket of  $G$  and the loop bracket of  $LM$ .

The commutative graded algebra  $\mathbb{H}_*(M)$  equipped with the trivial operator  $B$  can be considered as a Batalin-Vilkovisky algebra. Since the trivial section  $s : M \hookrightarrow LM$  is  $S^1$ -equivariant with respect to the



trivial action on  $M$ ,  $\mathbb{H}_*(s) : \mathbb{H}_*(M) \hookrightarrow \mathbb{H}_*(LM)$  is an inclusion of Batalin-Vilkovisky algebras and so an inclusion of Lie algebras.

Denote by  $\tilde{f} \in \pi_n(\Omega G)$ , the adjoint of  $f \in \pi_{n+1}(G)$ . By Lemma 28,  $\sigma_* \circ \text{hur}_{\Omega G} \tilde{f} = \text{hur}_G f$ . Since  $n \geq 1$ ,  $a := \text{hur}_{\Omega G} \tilde{f}$  is primitive and so by Corollary 26,  $\{\text{hur}_{\Omega G} \tilde{f} \otimes [M], 1 \otimes x\} = (-1)^n 1 \otimes \text{hur}_G f.x$ . Since by Theorem 15,  $H_*(\Theta_{G,M}) : H_*(\Omega G) \otimes \mathbb{H}_*(M) \rightarrow \mathbb{H}_*(LM)$  is a morphism of Lie algebras, finally we have

$$\{\tilde{\Gamma}_1(f), \mathbb{H}_*(s)(x)\} = (-1)^n \mathbb{H}_*(s)(\text{hur}_G f.x).$$

□

**Theorem 34.** *If  $\mathbb{k}$  is a field of characteristic 0 and  $G$  is  $\text{aut}_1 M$ , the monoid of self-equivalences homotopic to the identity, the morphism of Lie algebras considered in Theorem 33 is injective.*

*Proof.* Felix and Thomas [5, Theorem 2] showed that for  $n \geq 1$ ,  $\tilde{\Gamma}_1 : \pi_{n+1}(\text{aut}_1 M) \otimes \mathbb{k} \hookrightarrow \mathbb{H}_n(LM)$  is injective. Since  $s$  is a section,  $\mathbb{H}_*(s) = \mathbb{H}_*(M) \hookrightarrow \mathbb{H}_*(LM)$  is also injective. Our morphism of Lie algebras coincides with  $\tilde{\Gamma}_1$  in positive degree and with  $\mathbb{H}_*(s)$  in non-positive degree. Therefore, we have proved the theorem. □

Here is an example due to Yves Felix showing that the Lie algebras considered in Theorem 34 are not abelian even for a very simple manifold  $M$ .

*Example 35.* Let  $M$  be the product of spheres  $S^3 \times S^7$ . The minimal Sullivan model of  $M$  is  $(\Lambda(x_3, y_7), 0)$  with  $x_3$  in (upper) degree 3 and  $y_7$  in (upper) degree 7. Consider  $\text{Der } \Lambda(x_3, y_7)$  the Lie algebra of derivation of  $\Lambda(x_3, y_7)$  decreasing the degree. Let  $\theta_1, \theta_2$  and  $\theta_3 \in \text{Der } \Lambda(x_3, y_7)$  given by  $\theta_1(x_3) = 1, \theta_1(y_7) = 0, \theta_2(x_3) = 0, \theta_2(y_7) = 1, \theta_3(x_3) = 0$  and  $\theta_3(y_7) = x_4$ . These three derivations form a basis of the graded vector space  $\text{Der } \Lambda(x_3, y_7)$ . As graded Lie algebras,  $\pi_{\geq 2}(\text{aut}_1 M) \otimes \mathbb{Q}$  is isomorphic to  $\text{Der } \Lambda(x_3, y_7)$ . Since  $\theta_2 = \theta_1 \circ \theta_3 = \{\theta_1, \theta_3\}$ , the Samelson bracket in  $\pi_{\geq 2}(\text{aut}_1 M) \otimes \mathbb{Q}$  is non trivial.

## 7. THE BATALIN-VILKOVISKY ALGEBRA $\mathbb{H}_*(LG; \mathbb{Q})$

**Theorem 36.** *Let  $G$  be a path-connected topological monoid with a homotopy inverse acting on a smooth compact oriented manifold  $M$ . Then the Batalin-Vilkovisky algebra  $H_*(\Omega G; \mathbb{Q}) \otimes \mathbb{H}_*(M; \mathbb{Q})$  is the tensor product  $\mathbb{Q}[\pi_1(G)] \otimes \Lambda(s^{-1}\pi_{\geq 2}(G) \otimes \mathbb{Q}) \otimes \mathbb{H}_*(M; \mathbb{Q})$  of the group ring on  $\pi_1(G)$ , of the free commutative graded algebra on  $\pi_{\geq 1}(\Omega G) \otimes \mathbb{Q} \cong s^{-1}\pi_{\geq 2}(G) \otimes \mathbb{Q}$  and of  $\mathbb{H}_*(M; \mathbb{Q})$  equipped with the intersection product.*

For any  $f \in \pi_1(G)$ ,  $f_1, \dots, f_r \in \pi_{\geq 2}(G) \otimes \mathbb{Q}$ ,  $r \geq 0$  and  $x \in \mathbb{H}_*(M; \mathbb{Q})$ ,

$$(37) \quad B_{\Omega G \times M}(f s^{-1} f_1 \dots s^{-1} f_r \otimes x) = B_{\Omega G}(f s^{-1} f_1 \dots s^{-1} f_r) \otimes x \\ + (-1)^{|f_1| + \dots + |f_r| + r} (f s^{-1} f_1 \dots s^{-1} f_r \otimes \text{hur}_G f \cdot x + \\ \sum_{i=1}^r (-1)^{(|f_i|+1)(|f_{i+1}|+\dots+|f_r|+r-i+1)} f s^{-1} f_1 \dots s^{-1} f_{i-1} s^{-1} f_{i+1} \dots s^{-1} f_r \otimes \text{hur}_G f_i \cdot x).$$

*Proof.* The isomorphism of algebras  $H_*(\Omega G) \cong \mathbb{Q}[\pi_1(G)] \otimes H_*(\Omega \tilde{G})$  given in the proof of Lemma 19 is in fact an isomorphism of Hopf algebras. Therefore  $f$  is a group-like element of  $H_*(\Omega G)$ . So by Corollary 26  $\{f \otimes [M], 1 \otimes x\} = f \otimes \text{hur}_G f \cdot x$ .

By Milnor-Moore [4, Theorem 21.5] and Cartan-Serre theorems, the Hurewicz morphism  $\pi_{\geq 1} \Omega G \otimes \mathbb{Q} \cong \pi_{\geq 1} \Omega \tilde{G} \otimes \mathbb{Q} \rightarrow H_*(\Omega \tilde{G}; \mathbb{Q})$  extends to an isomorphism of Hopf algebras  $\Lambda(\pi_{\geq 1} \Omega G \otimes \mathbb{Q}) \xrightarrow{\cong} H_*(\Omega \tilde{G}; \mathbb{Q})$ . Denote by  $s^{-1} f_i \in \pi_{|f_i|-1}(\Omega G) \otimes \mathbb{Q}$  the adjoint of  $f_i \in \pi_{|f_i|}(G) \otimes \mathbb{Q}$ . As we already saw in Theorem 33, by Corollary 26 and Lemma 28,  $\{s^{-1} f_i \otimes [M], 1 \otimes x\} = (-1)^{|s^{-1} f_i|} 1 \otimes \text{hur}_G f_i \cdot x$ . Using the graded commutativity of the product and the graded antisymmetry of the Lie bracket of degree +1, the Poisson relation (7) can be rewritten as

$$\{bc, a\} = b\{c, a\} + (-1)^{|b||c|} c\{b, a\}$$

in any Gerstenhaber algebra. Note that the sign in this formula is given exactly by the Koszul rule. Therefore by immediate induction,

$$\{f s^{-1} f_1 \dots s^{-1} f_r \otimes [M], 1 \otimes x\} = \{f \otimes [M], s^{-1} f_1 \otimes [M] \dots s^{-1} f_r \otimes [M], 1 \otimes x\} \\ = s^{-1} f_1 \dots s^{-1} f_r \otimes [M] \cdot \{f \otimes [M], 1 \otimes x\} \\ + \sum_{i=1}^r (-1)^{|s^{-1} f_i| |s^{-1} f_{i+1}| \dots |s^{-1} f_r|} f s^{-1} f_1 \dots s^{-1} f_{i-1} s^{-1} f_{i+1} \dots s^{-1} f_r \otimes [M] \cdot \{s^{-1} f_i \otimes [M], 1 \otimes x\} \\ = f s^{-1} f_1 \dots s^{-1} f_r \otimes \text{hur}_G f \cdot x \\ + \sum_{i=1}^r (-1)^{|s^{-1} f_i| |s^{-1} f_{i+1}| \dots |s^{-1} f_r| + |s^{-1} f_i|} f s^{-1} f_1 \dots s^{-1} f_{i-1} s^{-1} f_{i+1} \dots s^{-1} f_r \otimes \text{hur}_G f_i \cdot x$$

Let  $a = f s^{-1} f_1 \dots s^{-1} f_r \in H_*(\Omega G)$ . By equation (27)

$$B_{\Omega G \times M}(a \otimes x) = B_{\Omega G}(a) \otimes x + (-1)^{|a|} \{a \otimes [M], 1 \otimes x\}$$

Therefore the theorem is proved.  $\square$

In the proof of Theorem 36, we have only used that the algebra  $H_*(\Omega G; \mathbb{Q})$  is generated by its spherical elements (i.e. elements in the image of the Hurewicz map). Therefore the formula (37) holds over any

principal ideal domain  $\mathbb{k}$  where  $H_*(\Omega G; \mathbb{k})$  is generated by its spherical elements. This is also the case for example if  $G$  is the circle  $S^1$  or the three-dimensional sphere  $S^3$ . This explains why formula (37) generalizes Propositions 29 and 30.

Theorem 36 tells us in particular that if we know the Batalin-Vilkovisky algebra  $H_*(\Omega G; \mathbb{Q})$  and the action of the spherical elements of  $H_*(G)$  on  $\mathbb{H}_*(M)$ , we can compute the Batalin-Vilkovisky algebra  $H_*(\Omega G; \mathbb{Q}) \otimes \mathbb{H}_*(M; \mathbb{Q})$ . In [6, Theorem 4.4], Gerald Gaudens and the author computed the Batalin-Vilkovisky algebra  $H_*(\Omega G; \mathbb{Q}) \cong H_*(\Omega^2 BG; \mathbb{Q})$  assuming that  $G$  is simply-connected. As we have already seen in Lemma 19, the Batalin-Vilkovisky algebra  $H_*(\Omega G)$  is also known when  $G$  is a path-connected Lie group. Therefore, we have:

**Corollary 38.** *Let  $G$  be a path-connected Lie group acting on a smooth compact oriented manifold  $M$ . Then the Batalin-Vilkovisky algebra  $H_*(\Omega G; \mathbb{Q}) \otimes \mathbb{H}_*(M; \mathbb{Q})$  is the tensor product  $\mathbb{Q}[\pi_1(G)] \otimes \Lambda(s^{-1}\pi_{\geq 3}(G) \otimes \mathbb{Q}) \otimes \mathbb{H}_*(M; \mathbb{Q})$  of the group ring on  $\pi_1(G)$ , of the symmetric algebra on  $s^{-1}\pi_{\geq 3}(G)$  and of  $\mathbb{H}_*(M; \mathbb{Q})$  equipped with the intersection product. For any  $f \in \pi_1(G)$ ,  $f_1, \dots, f_r \in \pi_{\geq 3}(G) \otimes \mathbb{Q}$ ,  $r \geq 0$  and  $x \in \mathbb{H}_*(M; \mathbb{Q})$ ,*

$$\begin{aligned} B_{\Omega G \times M}(f s^{-1} f_1 \dots s^{-1} f_r \otimes x) &= f s^{-1} f_1 \dots s^{-1} f_r \otimes \text{hur } f \cdot x \\ &+ \sum_{i=1}^r f s^{-1} f_1 \dots s^{-1} f_{i-1} s^{-1} f_{i+1} \dots s^{-1} f_r \otimes \text{hur } f_i \cdot x. \end{aligned}$$

Note that there are no signs in this Corollary.

*Proof.* The rational homotopy groups  $\pi_*(G) \otimes \mathbb{Q}$  are concentrated in odd degrees. Therefore  $s^{-1}f_i$  are all in even degree and the Corollary follows from Theorem 36.  $\square$

**Theorem 39.** *Let  $G$  be a path-connected compact Lie group. Let  $x_1, \dots, x_l$  be a basis of the free part of  $\pi_1(G)$ . Let  $x_{l+1}, \dots, x_r$  be a basis of the  $\mathbb{Q}$ -vector space  $\pi_{\geq 3}(G) \otimes \mathbb{Q}$ . Denote by  $(x_i^\vee)_{1 \leq i \leq r}$  the dual basis in  $(\pi_*(G) \otimes \mathbb{Q})^\vee$ . Then the Batalin-Vilkovisky algebra  $\mathbb{H}_*(LG; \mathbb{Q})$  is isomorphic to the tensor product  $\mathbb{Q}[\pi_1(G)] \otimes \Lambda(s^{-1}x_j)_{l < j \leq r} \otimes \Lambda(x_i^\vee)_{1 \leq i \leq r}$  of the group ring on  $\pi_1(G)$ , the polynomial algebra on  $s^{-1}x_{l+1}, \dots, s^{-1}x_r$  and the exterior algebra on  $x_1^\vee, \dots, x_r^\vee$ . For  $n_1, \dots, n_l \in \mathbb{Z}$  and  $n_{l+1}, \dots, n_r \geq 0$ ,  $p \geq 0$  and  $1 \leq j_1 < \dots < j_p \leq r$  and  $y$  any element of  $\pi_1(G)$  with torsion,*

$$\begin{aligned} &B(x_1^{n_1} \dots x_l^{n_l} y s^{-1} x_{l+1}^{n_{l+1}} \dots s^{-1} x_r^{n_r} \otimes x_{j_1}^\vee \dots x_{j_p}^\vee) \\ &= \sum_{i=1, j_i \leq l}^p (-1)^{i-1} n_{j_i} x_1^{n_1} \dots x_l^{n_l} y s^{-1} x_{l+1}^{n_{l+1}} \dots s^{-1} x_r^{n_r} \otimes x_{j_1}^\vee \dots \widehat{x_{j_i}^\vee} \dots x_{j_p}^\vee \end{aligned}$$

$$+ \sum_{i=1, j_i > l}^p (-1)^{i-1} n_{j_i} x_1^{n_1} \dots x_l^{n_l} y s^{-1} x_{l+1}^{n_{l+1}} \dots s^{-1} x_{j_i}^{n_{j_i}-1} \dots s^{-1} x_r^{n_r} \otimes x_{j_1}^\vee \dots \widehat{x_{j_i}^\vee} \dots x_{j_p}^\vee$$

Here  $\widehat{\phantom{x}}$  denotes omission.

When  $G = SO(n)$ ,  $n \geq 3$ , the special orthogonal group, the Batalin-Vilkovisky algebra  $\mathbb{H}_*(LSO(n); \mathbb{Q})$  has been first computed by Hepworth [10, Theorem 4]. but his formula uses a different presentation of the algebra  $H_*(\Omega SO(n); \mathbb{Q})$  by generators and relations.

*Proof.* As we already explained in Remark 16,

$$\mathbb{H}_*(\Theta_{G,G}) : H_*(\Omega G) \otimes \mathbb{H}_*(G) \xrightarrow{\cong} \mathbb{H}_*(LG)$$

is an isomorphism of Batalin-Vilkovisky algebras. By Milnor-Moore theorem, as Hopf algebras,  $H_*(G)$  is the exterior algebra  $\Lambda(x_i)_{1 \leq i \leq r}$  where  $x_i$  are primitive elements of odd degree. Therefore  $H^*(G)$  is the exterior algebra  $\Lambda(x_i^\vee)_{1 \leq i \leq r}$ .

By Poincaré duality, the cap product with the fundamental class  $[G]$  gives an isomorphism of graded algebras  $-\cap [G] : H^*(G) \xrightarrow{\cong} \mathbb{H}_*(G)$  between the cup product and the intersection product. (Note that this isomorphism respects degrees since a non-negative upper degree corresponds to a non-positive lower degree by the classical convention of [4, p. 41-2]).

The fundamental class of  $G$ ,  $[G]$ , is the product  $x_1 \dots x_r$ . Let  $i_1, \dots, i_p$  be  $p$  integers between 1 and  $r$ . Let  $1 \leq k \leq p$ . The cap product of  $x_{i_k}^\vee$  with  $x_{i_1} \dots x_{i_p}$  is  $(-1)^{k-1} x_{i_k}^\vee (x_{i_k}) x_{i_1} \dots \widehat{x_{i_k}} \dots x_{i_p}$  since  $(-1)^{k-1}$  is the Koszul sign obtained by exchanging  $x_{i_k}$  and  $x_{i_1} \dots x_{i_{k-1}}$ . Therefore

$$x_{i_k}^\vee \cap x_{i_1} \dots x_{i_p} = (-1)^{k-1} x_{i_1} \dots \widehat{x_{i_k}} \dots x_{i_p}.$$

In particular,

$$\begin{aligned} x_{j_p}^\vee \cap x_1 \dots x_r &= (-1)^{j_p-1} x_1 \dots \widehat{x_{j_p}} \dots x_r, \\ x_{j_{p-1}}^\vee \cap x_1 \dots \widehat{x_{j_p}} \dots x_r &= (-1)^{j_{p-1}-1} x_1 \dots \widehat{x_{j_{p-1}}} \widehat{x_{j_p}} \dots x_r, \\ &\vdots \\ &\vdots \end{aligned}$$

$$\text{and } x_{j_1}^\vee \cap x_1 \dots \widehat{x_{j_2}} \dots \widehat{x_{j_p}} \dots x_r = (-1)^{j_1-1} x_1 \dots \widehat{x_{j_1}} \dots \widehat{x_{j_p}} \dots x_r.$$

Therefore

$$\begin{aligned} x_{j_1}^\vee \dots x_{j_p}^\vee \cap [G] &= x_{j_1}^\vee \dots x_{j_p}^\vee \cap x_1 \dots x_r = x_{j_1}^\vee \cap (x_{j_2}^\vee \cap \dots (x_{j_p}^\vee \cap x_1 \dots x_r)) \\ &= (-1)^{j_1-1+\dots+j_p-1} x_1 \dots \widehat{x_{j_1}} \dots \widehat{x_{j_p}} \dots x_r. \end{aligned}$$

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Using the Poincaré duality isomorphism, we can now transport the action of  $H_*(G)$  on  $\mathbb{H}_*(G)$  given by multiplication into an action of  $H_*(G)$  on  $H^*(G)$ . Since

$$\begin{aligned} x_{j_i} \cdot \left( x_{j_1}^\vee \dots x_{j_p}^\vee \cap [G] \right) &= (-1)^{j_1-1+\dots+j_p-1} x_{j_i} x_1 \dots \widehat{x_{j_1}} \dots \widehat{x_{j_p}} \dots x_r \\ &= (-1)^{j_i-1-i+1} (-1)^{j_1-1+\dots+j_p-1} x_1 \dots \widehat{x_{j_1}} \dots \widehat{x_{j_{i-1}}} x_{j_i} \widehat{x_{j_{i+1}}} \dots \widehat{x_{j_p}} \dots x_r \\ &= (-1)^{i-1} x_{j_1}^\vee \dots \widehat{x_{j_i}^\vee} \dots x_{j_p}^\vee \cap [G], \end{aligned}$$

we obtain that

$$(40) \quad \begin{aligned} x_{j_i} \cdot x_{j_1}^\vee \dots x_{j_p}^\vee &= (-1)^{i-1} x_{j_1}^\vee \dots \widehat{x_{j_i}^\vee} \dots x_{j_p}^\vee \\ \text{and } x_j \cdot x_{j_1}^\vee \dots x_{j_p}^\vee &= 0 \text{ if } j \notin \{j_1, \dots, j_p\}. \end{aligned}$$

Let  $f$  be any element of  $\pi_1(G)$ . Then written multiplicatively,  $f$  is of the form  $x_1^{n_1} \dots x_l^{n_l} y$  for some  $n_1, \dots, n_l \in \mathbb{Z}$  and some element  $y$  of  $\pi_1(G)$  with torsion. Then  $hur_G : \pi_1(G) \rightarrow \pi_1(G) \otimes \mathbb{Q} \cong H_1(G; \mathbb{Q})$  maps  $f$  to  $n_1 x_1 + \dots + n_l x_l + 0$  while  $hur_G : \pi_{\geq 3}(G) \otimes \mathbb{Q} \rightarrow H_*(G; \mathbb{Q})$  maps  $x_j$  to itself for any  $l < j \leq r$ .

Therefore, by Corollary 38,  $H_*(\Omega G; \mathbb{Q}) \otimes \mathbb{H}_*(G; \mathbb{Q})$  is the tensor product of algebras  $\mathbb{Q}[\pi_1(G)] \otimes \Lambda(s^{-1} \pi_{\geq 3}(G) \otimes \mathbb{Q}) \otimes H^*(G; \mathbb{Q})$  and

$$\begin{aligned} B_{\Omega G \times G}(f s^{-1} x_{l+1}^{n_{l+1}} \dots s^{-1} x_r^{n_r} \otimes x_{j_1}^\vee \dots x_{j_p}^\vee) \\ = f s^{-1} x_{l+1}^{n_{l+1}} \dots s^{-1} x_r^{n_r} \otimes (n_1 x_1 + \dots + n_l x_l) \cdot x_{j_1}^\vee \dots x_{j_p}^\vee \\ + \sum_{j=l+1}^r n_j f s^{-1} x_{l+1}^{n_{l+1}} \dots s^{-1} x_j^{n_j-1} \dots s^{-1} x_r^{n_r} \otimes x_j \cdot x_{j_1}^\vee \dots x_{j_p}^\vee. \end{aligned}$$

Using (40), the theorem is proved.  $\square$

In [13, Theorems 10 and 12], we computed the Batalin-Vilkovisky algebra  $\mathbb{H}_*(LS^{2i+1})$  for all odd dimensional spheres. Using this computation, the previous Theorem can be given the following simple interpretation.

**Theorem 41.** *Let  $G$  be a path-connected compact Lie group. Then the Chas-Sullivan Batalin-Vilkovisky algebra on the rational free loop space homology on  $G$ ,  $\mathbb{H}_*(LG; \mathbb{Q})$  is isomorphic to the tensor product (in the sense of Example 8)*

$$\mathbb{Q}[\pi_1(G)_{tor}] \otimes \bigotimes_{k=0}^{+\infty} \mathbb{H}_*(LS^{2k+1}; \mathbb{Q})^{\otimes \dim \pi_{2k+1}(G) \otimes \mathbb{Q}}.$$

Here  $\mathbb{Q}[\pi_1(G)_{tor}]$  denotes the Batalin-Vilkovisky algebra with trivial operator whose underlying algebra is the group ring on the torsion subgroup of  $\pi_1(G)$ .

Note that this theorem extends the case of  $G = SU(n+1)$  the special unitary group first proved by Tamanoi in [18, Corollary C] using a different method.

*Proof.* Recall from [13, Theorem 10] that  $\mathbb{H}_*(LS^1) \cong \mathbb{Q}[\mathbb{Z}] \otimes \Lambda x^\vee$  with

$$B(x^n \otimes x^\vee) = n(x^n \otimes 1), \quad B(x^n \otimes 1) = 0$$

for all  $n \in \mathbb{Z}$ . Here  $x$  denotes a generator of  $\pi_1(S^1) \cong \mathbb{Z}$ . Recall from [13, Theorem 16] that for  $i \geq 1$ ,  $\mathbb{H}_*(LS^{2i+1}) \cong \Lambda(s^{-1}x) \otimes \Lambda x^\vee$ , with  $B(s^{-1}x^n \otimes x^\vee) = n(s^{-1}x^{n-1} \otimes 1)$  and  $B(s^{-1}x^n \otimes 1) = 0$  for all  $n \geq 0$ . Here  $x$  denote a generator of lower degree  $2i+1$ .

Let  $x_i$ 's be the generators defined by Theorem 39. Let  $\Theta$  be the isomorphism of algebras from

$$\mathbb{H}_*(LS^1; \mathbb{Q})^{\otimes \dim \pi_1(G) \otimes \mathbb{Q}} \otimes \mathbb{Q}[\pi_1(G)_{\text{tor}}] \otimes \bigotimes_{k=1}^{+\infty} \mathbb{H}_*(LS^{2k+1}; \mathbb{Q})^{\otimes \dim \pi_{2k+1}(G) \otimes \mathbb{Q}}$$

to

$$\mathbb{Q}[\pi_1(G)] \otimes \Lambda(s^{-1}x_j)_{l < j \leq r} \otimes \Lambda(x_i^\vee)_{1 \leq i \leq r}$$

mapping

- for  $1 \leq i \leq l$ , the elements  $x_i^{n_i} \otimes 1$  (respectively  $x_i^{n_i} \otimes x_i^\vee$ ) in the  $i^{\text{th}}$  factor of  $\mathbb{H}_*(LS^1; \mathbb{Q})^{\otimes \dim \pi_1(G) \otimes \mathbb{Q}}$  to  $x_i^{n_i} \otimes 1 \otimes 1$  (respectively  $x_i^{n_i} \otimes 1 \otimes x_i^\vee$ ),
- for each  $y \in \pi_1(G)_{\text{tor}}$ , the element  $y$  to  $y \otimes 1 \otimes 1$  and
- for  $l < i \leq r$ , the elements  $s^{-1}x_i^{n_i} \otimes 1$  (respectively  $s^{-1}x_i^{n_i} \otimes x_i^\vee$ ) in the  $(i-l)^{\text{th}}$  factor of  $\bigotimes_{k=1}^{+\infty} \mathbb{H}_*(LS^{2k+1}; \mathbb{Q})^{\otimes \dim \pi_{2k+1}(G) \otimes \mathbb{Q}}$  to  $1 \otimes s^{-1}x_i^{n_i} \otimes 1$  (respectively  $1 \otimes s^{-1}x_i^{n_i} \otimes x_i^\vee$ ).

Explicitly  $\Theta$  is the linear isomorphism mapping the element

$$(x_1^{n_1} \otimes 1) \otimes \cdots \otimes (x_{j_1}^{n_{j_1}} \otimes x_{j_1}^\vee) \otimes \cdots \otimes (x_l^{n_l} \otimes 1) \otimes y \\ \otimes (s^{-1}x_{l+1}^{n_{l+1}} \otimes 1) \otimes \cdots \otimes (s^{-1}x_{j_p}^{n_{j_p}} \otimes x_{j_p}^\vee) \otimes \cdots \otimes (s^{-1}x_r^{n_r} \otimes 1)$$

to  $x_1^{n_1} \cdots x_l^{n_l} y \otimes s^{-1}x_{l+1}^{n_{l+1}} \cdots s^{-1}x_r^{n_r} \otimes x_{j_1}^\vee \cdots x_{j_p}^\vee$ .

In the tensor product of Batalin-Vilkovisky algebras

$$\mathbb{H}_*(LS^1; \mathbb{Q})^{\otimes \dim \pi_1(G) \otimes \mathbb{Q}} \otimes \mathbb{Q}[\pi_1(G)_{\text{tor}}] \otimes \bigotimes_{i=1}^{+\infty} \mathbb{H}_*(LS^{2i+1}; \mathbb{Q})^{\otimes \dim \pi_{2i+1}(G) \otimes \mathbb{Q}},$$

the operator  $B$  is given by

$$B(x_1^{n_1} \otimes 1 \otimes \cdots \otimes x_{j_1}^{n_{j_1}} \otimes x_{j_1}^\vee \otimes \cdots \otimes x_l^{n_l} \otimes 1 \otimes y \\ \otimes s^{-1}x_{l+1}^{n_{l+1}} \otimes 1 \otimes \cdots \otimes s^{-1}x_{j_p}^{n_{j_p}} \otimes x_{j_p}^\vee \otimes \cdots \otimes s^{-1}x_r^{n_r} \otimes 1)$$

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$$\begin{aligned}
&= \sum_{i=1, j_i \leq l}^p (-1)^{i-1} x_1^{n_1} \otimes 1 \otimes \cdots \otimes x_{j_1}^{n_{j_1}} \otimes x_{j_1}^\vee \otimes \cdots \otimes B(x_{j_i}^{n_{j_i}} \otimes x_{j_i}^\vee) \otimes \cdots \otimes x_l^{n_l} \otimes 1 \otimes y \\
&\quad \otimes s^{-1} x_{l+1}^{n_{l+1}} \otimes 1 \otimes \cdots \otimes s^{-1} x_{j_p}^{n_{j_p}} \otimes x_{j_p}^\vee \otimes \cdots \otimes s^{-1} x_r^{n_r} \otimes 1 \\
&+ \sum_{i=1, j_i > l}^p (-1)^{i-1} x_1^{n_1} \otimes 1 \otimes \cdots \otimes x_{j_1}^{n_{j_1}} \otimes x_{j_1}^\vee \otimes \cdots \otimes x_l^{n_l} \otimes 1 \otimes y \\
&\otimes s^{-1} x_{l+1}^{n_{l+1}} \otimes 1 \otimes \cdots \otimes B(s^{-1} x_{j_i}^{n_{j_i}} \otimes x_{j_i}^\vee) \otimes \cdots \otimes s^{-1} x_{j_p}^{n_{j_p}} \otimes x_{j_p}^\vee \otimes \cdots \otimes s^{-1} x_r^{n_r} \otimes 1 \\
&= \sum_{i=1, j_i \leq l}^p (-1)^{i-1} x_1^{n_1} \otimes 1 \otimes \cdots \otimes x_{j_1}^{n_{j_1}} \otimes x_{j_1}^\vee \otimes \cdots \otimes (n_{j_i} x_{j_i}^{n_{j_i}} \otimes 1) \otimes \cdots \otimes x_l^{n_l} \otimes 1 \otimes y \\
&\quad \otimes s^{-1} x_{l+1}^{n_{l+1}} \otimes 1 \otimes \cdots \otimes s^{-1} x_{j_p}^{n_{j_p}} \otimes x_{j_p}^\vee \otimes \cdots \otimes s^{-1} x_r^{n_r} \otimes 1 \\
&+ \sum_{i=1, j_i > l}^p (-1)^{i-1} x_1^{n_1} \otimes 1 \otimes \cdots \otimes x_{j_1}^{n_{j_1}} \otimes x_{j_1}^\vee \otimes \cdots \otimes x_l^{n_l} \otimes 1 \otimes y \\
&\otimes s^{-1} x_{l+1}^{n_{l+1}} \otimes 1 \otimes \cdots \otimes (n_{j_i} s^{-1} x_{j_i}^{n_{j_i}-1} \otimes 1) \otimes \cdots \otimes s^{-1} x_{j_p}^{n_{j_p}} \otimes x_{j_p}^\vee \otimes \cdots \otimes s^{-1} x_r^{n_r} \otimes 1
\end{aligned}$$

Therefore

$$\Theta \circ B \circ \Theta^{-1} (x_1^{n_1} \cdots x_l^{n_l} y \otimes s^{-1} x_{l+1}^{n_{l+1}} \cdots s^{-1} x_r^{n_r} \otimes x_{j_1}^\vee \cdots x_{j_p}^\vee)$$

is given by Theorem 39.  $\square$

8. SEMI-DIRECT PRODUCT UP TO HOMOTOPY

This section on split fibrations is needed in Section 3. A split short exact sequences of groups gives a semi-direct product. The following Proposition, which is certainly not new, is a homotopy version of this fact.

**Proposition 42.** *Let  $F \xrightarrow{j} E \xrightarrow{p} B$  be a fibration with an homotopy section  $\sigma : B \rightarrow E$ ,  $p \circ \sigma \approx id_E$ , such that  $j$ ,  $p$  and  $\sigma$  are morphisms of  $H$ -groups. Then  $j$  admits a retract  $r : E \rightarrow F$  up to homotopy such that*

- *the composite  $r \circ \sigma$  is homotopically trivial and*
- *the composite  $F \times B \xrightarrow{j \times \sigma} E \times E \xrightarrow{\mu_E} E$  and  $(r, p) : E \rightarrow F \times B$  are homotopy inverse.*

*Proof.* Let  $Z$  be any pointed space. Applying the pointed homotopy class functor  $[Z, -]$ , the Nomura-Puppe long exact sequence gives the short exact sequence of groups with the splitting  $[Z, \sigma]$ ,

$$1 \rightarrow [Z, F] \xrightarrow{[Z, j]} [Z, E] \xrightarrow{[Z, p]} [Z, B] \rightarrow 1.$$

Therefore the group  $[Z, E]$  is isomorphic to the semi-direct product of groups  $[Z, F] \rtimes [Z, B]$ . More precisely:

The map  $E \rightarrow E$  sending  $e \mapsto e \cdot \sigma \circ p(e)^{-1}$  induces the map

$$[Z, \mu_E \circ (id_E, inv \circ \sigma \circ p)] : [Z, E] \rightarrow [Z, E].$$

Let  $R : [Z, E] \rightarrow [Z, F]$  be the unique map such that the composite  $[Z, j] \circ R$  coincides with  $[Z, \mu_E \circ (id_E, inv \circ \sigma \circ p)]$ . The map  $R$  is a retract of  $[Z, j]$ .

The bijection  $[Z, F] \times [Z, B] = [Z, F \times B] \rightarrow [Z, E]$  is given by  $\mu_{[Z, E]} \circ ([Z, j] \times [Z, \sigma]) = [Z, \mu_E \circ (j \times \sigma)]$ . Its inverse  $[Z, E] \rightarrow [Z, F] \times [Z, B]$  is the map  $(R, [Z, p])$ . By Yoneda lemma, the natural transformation  $R : [Z, E] \rightarrow [Z, F]$  can be uniquely written as a  $[Z, r]$  where  $r$  is a pointed map from  $E$  to  $F$ . By Yoneda Lemma again, the composite  $F \times B \xrightarrow{j \times \sigma} E \times E \xrightarrow{\mu_E} E$  and  $(r, p) : E \rightarrow F \times B$  are homotopy inverse,  $r \circ j \approx Id_F$  and  $r \circ \sigma$  is homotopically trivial.  $\square$

## REFERENCES

- [1] M. Chas and D. Sullivan, *String topology*, preprint: math.GT/991159, 1999.
- [2] D. Chataur and L. Menichi, *String topology of classifying spaces*, preprint: math.AT/0801.0174v3, 2008.
- [3] F. Cohen, T. Lada, and J. May, *The homology of iterated loop spaces*, Lecture Notes in Mathematics, vol. 533, Springer-Verlag, 1976.
- [4] Y. Félix, S. Halperin, and J.-C. Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics, vol. 205, Springer-Verlag, 2000.
- [5] Y. Félix and J.-C. Thomas, *Monoid of self-equivalences and free loop spaces*, Proc. Amer. Math. Soc. **132** (2004), no. 1, 305–312.
- [6] G. Gaudens and L. Menichi, *Batalin-Vilkovisky algebras and the J-homomorphism*, Topology Appl. **156** (2008), no. 2, 365–374.
- [7] M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. of Math. **78** (1963), no. 2, 267–288.
- [8] E. Getzler, *Batalin-Vilkovisky algebras and two-dimensional topological field theories*, Comm. Math. Phys. **159** (1994), no. 2, 265–285.
- [9] K. Gruher and P. Salvatore, *Generalized string topology operations*, Proc. London Math. Soc. (3) **96** (2008), no. 1, 78–106.
- [10] Richard A. Hepworth, *String topology for Lie groups*, J. Topol. **3** (2010), no. 2, 424–442.
- [11] P. J. Hilton and U. Stammback, *A course in homological algebra*, second ed., Graduate Texts in Mathematics, vol. 4, Springer-Verlag, New York, 1997.
- [12] Jean-Louis Koszul, *Crochet de Schouten-Nijenhuis et cohomologie*, Astérisque (1985), no. Numero Hors Serie, 257–271, The mathematical heritage of Élie Cartan (Lyon, 1984).
- [13] L. Menichi, *String topology for spheres*, Comment. Math. Helv. **84** (2009), no. 1, 135–157.



## A BATALIN-VILKOVISKY ALGEBRA MORPHISM FROM DOUBLE LOOP SPACES TO FREE

- [14] J. Milnor, *Morse theory*, Based on lecture notes by M. Spivak and R. Wells. Annals of Mathematics Studies, No. 51, Princeton University Press, Princeton, N.J., 1963.
- [15] John Milnor, *On spaces having the homotopy type of CW-complex*, Trans. Amer. Math. Soc. **90** (1959), 272–280.
- [16] Michael Penkava and Albert Schwarz, *On some algebraic structures arising in string theory*, Perspectives in mathematical physics, Conf. Proc. Lecture Notes Math. Phys., III, Int. Press, Cambridge, MA, 1994, pp. 219–227.
- [17] Edwin H. Spanier, *Algebraic topology*, Springer-Verlag, New York, 1981, Corrected reprint.
- [18] H. Tamanoi, *Batalin-Vilkovisky Lie algebra structure on the loop homology of complex Stiefel manifolds*, Int. Math. Res. Not. (2006), 1–23.
- [19] C. Weibel, *An introduction to homological algebra*, Cambridge University Press, 1994.
- [20] G. Whitehead, *Elements of homotopy theory*, Graduate Texts in Mathematics, vol. 61, Springer-Verlag, 1978.

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