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p-th powers in mod p cohomology of fibers

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Abstract. Let k be a non-negative integer. Let $F \hookrightarrow E \twoheadrightarrow B$ be a fibration whose base space B is a finite simply-connected CW-complex of dimension $\leqslant p^k$ and whose total space E is a path-connected CW-complex of dimension $\leqslant p^k - 1$. If $\alpha \in H^+(F;\mathbb{F}_p)$ then $\alpha^{p^k} = 0$. © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Les puissances p-èmes dans la cohomologie modulo p de fibres

Résumé. *Soit $k \in \mathbb{N}^*$. Considérons une fibration $F \hookrightarrow E \twoheadrightarrow B$ dont la base B est un CW-complexe fini simplement connexe de dimension $\leqslant p^k$, et dont l'espace total E est un CW-complexe fini connexe par arcs de dimension $\leqslant p^k - 1$. Si $\alpha \in H^+(F;\mathbb{F}_p)$ alors $\alpha^{p^k} = 0$.* © 2001 Académie des sciences/Éditions scientifiques et médicales Elsevier SAS

Version française abrégée

Soit p un nombre premier quelconque. Nous notons $H^*(X) = \mathbb{F}_p \oplus H^+(X)$ la cohomologie de l'espace connexe par arcs X à coefficients dans le corps premier $\mathbb{F}_p$. Nous étudions les éléments $(\alpha^{p^k}, \alpha \in H^+(X), k \geqslant 1)$, appelés puissances p^k-èmes, de l'algèbre de cohomologie d'espaces X obtenues comme produit fibré (homotopique) de CW-complexes finis simplement connexes. Nous démontrons :

THÉORÈME A. – *Soient $r, k \in \mathbb{N}^*$. Considérons un produit fibré d'espaces* :

$$\begin{array}{ccc} E \times_B X & \longrightarrow & E \\ \downarrow & & \downarrow \pi \\ X & \longrightarrow & B \end{array}$$

où

- *π est une fibration de Serre*;
- *B est un CW-complexe fini r-connexe de dimension inférieure ou égale à rp^k* ;
- *E et X sont deux CW-complexes finis $(r-1)$-connexes tels que $E \times X$ soit de dimension inférieure ou égale à $rp^k - 1$.*

Alors les puissances p^k-èmes sont nulles dans $H^+(E \times_B X)$.

Note présentée par Henri CARTAN.

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Dans le cas particulier de l'espace B^{S^1} des lacets libres sur un CW-complexe B (*voir* version anglaise), nous pouvons minimiser les hypothèses du théorème A :

THÉORÈME B. – *Soient $r, k \in \mathbb{N}^*$. Si B est un CW-complexe fini r-connexe de dimension inférieure ou égale à rp^k alors les puissances p^k-èmes s'annulent dans $H^+(B^{S^1})$.*

THÉORÈME C (À comparer avec [8, 10.8]). – *Soient $r, k \in \mathbb{N}^*$. Soit $F \overset{j}{\hookrightarrow} E \overset{\pi}{\twoheadrightarrow} B$ une fibration de Serre d'espace total E connexe par arcs. Si la base B est un CW-complexe fini r-connexe de dimension inférieure ou égale à rp^k alors pour tout $\alpha \in H^*(F)$, $\alpha^{p^k} \in \operatorname{Im} H^*(j)$.*

En Corollaire du théorème A ou du théorème C, nous obtenons

COROLLARY. – *Soit B un CW-complexe fini simplement connexe. Pour tout $\alpha \in H^+(\Omega B)$, il existe $k \in \mathbb{N}^*$ telle que $\alpha^{p^k} = 0$.*

Signalons que ce corollaire résulte aussi du théorème suivant démontré par Lannes et Schwartz en utilisant les opérations de Steenrod dans la suite spectrale d'Eilenberg–Moore.

THÉORÈME [5, proposition 0.6]. – *Soit B un CW-complexe simplement connexe ayant un nombre fini de cellules en chaque dimension. Si l'algèbre de Steenrod agit sur $H^*(B)$ avec des orbites finies, alors elle agit aussi sur $H^*(\Omega B)$ avec des orbites finies.*

We work over the prime field $\mathbb{F}_p$ with p an odd or even prime. The homology and cohomology of spaces are considered with coefficients in $\mathbb{F}_p$.

In [1], Anick proved using algebraic models:

THEOREM [1, 9.1]. – *Let r be a non-negative integer. Let B be a simply-connected space with a finite type homology concentrated in degrees $i \in [r+1, rp]$. Then all p-th powers vanish in $H^+(\Omega B)$.*

This result was suggested by McGibbon and Wilkerson [7, p. 699]. The aim of this Note is to give two different generalisations of Anick theorem: Theorem A and Theorem C below.

The first one, whose proof is inspired by the proof of a result of Lannes and Schwartz [5, Proposition 0.6], uses the (vertical) Steenrod operations in the Eilenberg–Moore spectral sequence:

THEOREM A. – *Let r and k be two non-negative integers. Consider a fiber product of spaces*:

$$\begin{tikzcd} E \times_B X \arrow[r] \arrow[d] & E \arrow[d, "\pi"] \\ X \arrow[r] & B \end{tikzcd}$$

where

- *π is a Serre fibration and*
- *$H^*(E)$, $H^*(X)$ and $H^*(B)$ are of finite type.*

If B is simply-connected with homology concentrated in degrees $i \in [r+1, rp^k]$, and the product space $E \times X$ is path connected with homology $H_*(E \times X)$ concentrated in degrees $i \in [r, rp^k - 1]$, then all p^k-th powers vanish in $H^+(E \times_B X)$.

Proof. – We suppose that p is an odd prime. The case $p = 2$ is similar. Let $\mathcal{A}$ denote the mod p Steenrod algebra. The degree of an element α is denoted $|\alpha|$. Recall from [9,11,12], that the Eilenberg–Moore spectral sequence is a strongly convergent second quadrant cohomological spectral sequence of $\mathcal{A}$-modules:

$$E_2^{-s,*} \cong \operatorname{Tor}^{-s,*}_{H^*(B)}\bigl(H^*(E), H^*(X)\bigr) \Rightarrow H^*(E \times_B X).$$

More precisely, there exists a convergent filtration of $\mathcal{A}$-modules on $H^*(E \times_B X)$:

$$H^*(E \times_B X) \supset \dots F_s \supset F_{s-1} \dots F_1 \supset F_0 \supset F_{-1} ==$$

THEOREM [3, 2.9(i)]. – *Let r and k be two non-negative integers. If B is a simply-connected space with a finite type homology concentrated in degrees $i \in [r+1, rp^k]$ then all p^k-th powers vanish in $H^+(\Omega B)$.*

This result generalizes in:

THEOREM C (Compare with [8, 10.8]). – *Let r and k be two non-negative integers. Let $F \overset{j}{\hookrightarrow} E \overset{\pi}{\twoheadrightarrow} B$ be a Serre fibration with E path connected. If B is a simply-connected space with finite type homology concentrated in degrees $i \in [r+1, rp^k]$ then, for any $\alpha \in H^*(F)$, $\alpha^{p^k} \in \operatorname{Im} H^*(j)$.*

Proof. – The proof follows the lines of [3, 2.9]. Since $H^{\leqslant r}(B) = 0$, $\alpha \in E_2^{0,*}$ survives till $E_{r+1}^{0,*}$. Therefore by a theorem of Araki [2] and Vázquez [13] (*see* also [10], Proposition 2.5, Case 2), $\alpha^{p^k} \in E_2^{0,*}$ survives till $E_{rp^k+1}^{0,*}$. Since $H^{>rp^k}(B) = 0$,

$$E_{rp^k+1}^{0,*} = E_\infty^{0,*} = \operatorname{Im} H^*(j).$$

□

In order to see that the hypothesis in the Félix–Halperin–Thomas theorem (and in Theorem B) cannot be improved, consider $B = \Sigma\mathbb{CP}^{p^k}$, the suspension of the p^k-dimension complex projective space.

Observe also that in Theorem C, α^{p^k} is not zero in general. Indeed, take π to be the fibration associated to the suspension of the Hopf map from S^{2p^k-1} to $\mathbb{CP}^{p^k-1}$ [8, Remark 9.9].

Finally, we remark that the following question of McGibbon and Wilkerson remains unsolved.

Question [7, p. 699] (See also [6], Section 9, Question 3). – Let B be a finite simply-connected CW-complex and p a prime large enough. Do all the Steenrod operations act trivially on $H^*(\Omega B)$?

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[1] To prove it, redo [9] using the cocyclic Cobar construction of Jones ([4], exemple 1.2) instead of the geometric Cobar construction.

References

[1] Anick D.J., Hopf algebras up to homotopy, J. Amer. Math. Soc. 2 (3) (1989) 417–453.
[2] Araki S., Steenrod reduced powers in the spectral sequences associated with a fibering I, II, Mem. Fac. Sci. Kyushu Univ. Ser. A 11 (1957) 15–64, 81–97.
[3] Félix Y., Halperin S., Thomas J.-C., The Serre spectral sequence of a multiplicative fibration, Prépublications, université d'Angers, 1998, http://math.univ-angers.fr/preprint.html.
[4] Jones J.D.S., Cyclic homology and equivariant homology, Invent. Math. 87 (2) (1987) 403–423.
[5] Lannes J., Schwartz L., A propos de conjectures de Serre et Sullivan, Invent. Math. 83 (1986) 593–603.
[6] McGibbon C.A., Phantom maps, in: James I.M. (Ed.), Handbook of Algebraic Topology, North-Holland, Amsterdam, 1995, pp. 1209–1257.
[7] McGibbon C.A., Wilkerson C.W., Loop spaces of finite complexes at large primes, Proc. Amer. Math. Soc. 96 (1986) 698–702.
[8] Menichi L., On the cohomology algebra of a fiber, Preprint Hopf Topology Archive, 1999.
[9] Rector D.L., Steenrod operations in the Eilenberg–Moore spectral sequence, Comment. Math. Helv. 45 (1970) 540–552.
[10] Sawka J., Odd primary Steenrod operations in first-quadrant spectral sequences, Trans. Am. Math. Soc. 273 (1982) 737–752.
[11] Schwartz L., Unstable modules and unstable algebras. Applications to homotopy theory, in: Lectures Notes of the Grenoble Summer School, 1997, http://www-fourier.ujf-grenoble.fr.
[12] Smith L., On the Künneth theorem. I: The Eilenberg–Moore spectral sequence, Math. Z. 116 (1970) 94–140.
[13] Vázquez Garcia R., Note on Steenrod squares in the spectral sequence of a fibre space, Bol. Soc. Math Mexicana 2 (1957) 1–8.