

# BATALIN-VILKOVISKY ALGEBRAS AND THE $J$ -HOMOMORPHISM

GERALD GAUDENS\* AND LUC MENICHI\*\*†

ABSTRACT. Let  $X$  be a topological space. The homology of the iterated loop space  $H_*\Omega^n X$  is an algebra over the homology of the framed  $n$ -disks operad  $H_*f\mathcal{D}_n$  [4, 9]. We explicitly determine this  $H_*f\mathcal{D}_n$ -algebra structure on  $H_*(\Omega^n X; \mathbb{Q})$ . We show that the action of  $H_*(SO(n))$  on the iterated loop space  $H_*\Omega^n X$  is related to the  $J$ -homomorphism and that the  $BV$ -operator on  $H_*(\Omega^2 X)$  vanishes on spherical classes only in characteristic other than 2.

## 1. INTRODUCTION

According to their importance in algebraic topology, iterated loop spaces have been studied for a long time: the structure of an  $n$ -fold loop space is essentially codified by an action the little  $n$ -disks operad  $\mathcal{D}_n$  [7]. Hence *via* the homology functor  $H_*$  over an arbitrary field  $\mathbb{K}$ , the homology of an  $n$ -fold loop space enjoys the rich structure of algebra over the operad  $H_*\mathcal{D}_n$ , as studied in [2]. The slightly larger operad of *framed*  $n$ -disks  $f\mathcal{D}_n$  acts naturally on  $n$ -fold loop spaces, yielding an action of the operad  $H_*f\mathcal{D}_n$  in homology with field coefficients, which for the case  $n = 2$  induces a *Batalin-Vilkovisky algebra structure* [4]. A systematic study of  $f\mathcal{D}_n$  spaces was initiated in [9], where their homology was partially analyzed. We shall in this work pursue somewhat this study. We completely characterize the  $H_*f\mathcal{D}_n$  structure of iterated loop spaces in characteristic zero under some connectivity assumptions. We notice for instance that the rational homology of the double loop space on a 2-connected space is the Batalin-Vilkovisky algebra *freely generated* by its rational homotopy Lie algebra (Theorem 4.4).

Our method relies on a connection between the  $f\mathcal{D}_n$  structure of iterated loop spaces and the classical  $J$ -homomorphism, together with a certain compatibility with the Hurewicz homomorphism. Although in characteristic 2 we cannot have a simple description as in the rational case (as follows from our last section), our methods can be used further to provide more computations as we will show in a forthcoming work.

Let us now give the content of this paper with more details.

Except when specified, the homology functor  $H_*$  is considered over an arbitrary field  $\mathbb{K}$ . Let  $n \geq 2$ . Let  $X$  be a pointed topological space. Consider the iterated loop space  $\Omega^n X$ . The action of the little  $n$ -disks operad  $\mathcal{D}_n$  on  $\Omega^n X$  gives  $H_*\Omega^n X$

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\*University of Bonn, Germany.

\*\*University of Angers, France.

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the structure of  $H_*\mathcal{D}_n$ -algebra. Fred Cohen [2] has shown that this equips  $H_*\Omega^n X$  with the structure of an  $e_n$ -algebra.

**Definition 1.1.** An  $e_n$ -algebra is a commutative graded algebra  $A$  equipped with a linear map  $\{-, -\} : A \otimes A \rightarrow A$  of degree  $n - 1$ , such that:

a) the bracket  $\{-, -\}$  gives  $A$  the structure of graded Lie algebra of degree  $n - 1$ . This means that for each  $a, b$  and  $c \in A$ ,

$$\{a, b\} = -(-1)^{(|a|+n-1)(|b|+n-1)}\{b, a\} \text{ and}$$

$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{(|a|+n-1)(|b|+n-1)}\{b, \{a, c\}\}.$$

b) the product and the Lie bracket satisfy the Poisson relation:

$$\{a, bc\} = \{a, b\}c + (-1)^{(|a|+n-1)|b|}b\{a, c\}.$$

Suppose that  $X$  is a based space with base point  $* \in X$  equipped with a pointed action of  $SO(n)$ : pointed action means that for any element  $g$  in  $SO(n)$ , we have  $g.* = *$ . Getzler [4], Salvatore and Wahl [9] have noticed that the little  $n$ -disks operad  $\mathcal{D}_n$  is an  $SO(n)$ -operad which acts in the category of  $SO(n)$ -spaces on  $\Omega^n X$ , or equivalently that the framed  $n$ -disks operad  $f\mathcal{D}_n$  acts on  $\Omega^n X$ . Therefore  $H_*\Omega^n X$  is an algebra over the operad  $H_*\mathcal{D}_n$  in the category of  $H_*SO(n)$ -module, or equivalently is an algebra over the operad  $H_*f\mathcal{D}_n$ . The first goal of this paper is to provide some explicit computations of this structure on  $H_*\Omega^n X$  for various  $X$  and various coefficients  $\mathbb{K}$ .

It is obvious that the structure of  $H_*f\mathcal{D}_n$ -algebra is the structure of an  $e_n$ -algebra together with the structure of  $H_*SO(n)$ -module which satisfy some compatibility relations. Over any coefficients  $\mathbb{K}$ , when  $n = 2$ , Getzler [4] has shown that an  $H_*(f\mathcal{D}_2; \mathbb{K})$ -algebra is a  $BV_2$ -algebra (i.e. Batalin-Vilkovisky algebra).

**Definition 1.2.** [9, Def 5.2] A  $BV_n$ -algebra  $A$  is an  $e_n$ -algebra with a linear endomorphism  $BV : A \rightarrow A$  of degree  $n - 1$  such that  $BV \circ BV = 0$  and for each  $a, b \in A$ ,

$$(1.3) \quad \{a, b\} = (-1)^{|a|} \left( BV(ab) - (BVa)b - (-1)^{|a|}a(BVb) \right).$$

The bracket measures the deviation of the operator  $BV$  from being a derivation with respect to the product. Furthermore, in a  $BV_n$ -algebra,  $BV$  satisfies the formula [4, Proposition 1.2] or [9, (7) in section 5]

$$(1.4) \quad BV[a, b] = [BVa, b] + (-1)^{|a|+1}[a, BVb].$$

Recall that  $H_*(SO(2k-1); \mathbb{Q})$  is an exterior algebra on generators  $a_{4i-1}$  for  $1 \leq i \leq k-1$  while  $H_*(SO(2k); \mathbb{Q})$  is isomorphic to  $H_*(SO(2k-1); \mathbb{Q})$  with an adjoined exterior generator  $a_{2k-1}$  [5, p. 300].

When  $\mathbb{K} = \mathbb{Q}$  and for any  $n \geq 2$ , Salvatore and Wahl have computed exactly what  $H_*(f\mathcal{D}_n; \mathbb{Q})$ -algebras are

**Theorem 1.5.** [9, Th. 5.4] *An  $H_*(f\mathcal{D}_{2k-1}; \mathbb{Q})$ -algebra  $A$  is an  $e_{2k-1}$ -algebra such that all the generators of the algebra  $H_*(SO(2k-1); \mathbb{Q})$  act on  $A$  as derivation with respect to the product. An  $H_*(f\mathcal{D}_{2k}; \mathbb{Q})$ -algebra  $A$  is a  $BV_{2k}$ -algebra such that the generators  $a_3, \dots, a_{4k-5}$  of the algebra  $H_*(SO(2k); \mathbb{Q})$  act on  $A$  as derivation with respect to the product and where the last generator  $a_{2k-1}$  defines the  $BV$ -operator.*

Our first remark (Proposition 2.1) is that for many pointed  $SO(n)$ -spaces  $X$ , this  $H_*(f\mathcal{D}_n; \mathbb{Q})$ -algebra structure on  $H_*(\Omega^n X; \mathbb{Q})$  reduces to:

- an  $e_n$ -algebra structure if  $n$  is odd,

- a  $BV_n$ -algebra structure if  $n$  is even.

That is, all the generators of  $H_*(SO(n); \mathbb{Q})$ , except when  $n$  is even the generator in degree  $n - 1$ , act trivially in cases of interest. Proposition 2.1 holds in particular for any pointed topological space  $X$  considered as a trivial pointed  $SO(n)$ -space. In the rest of this paper, this is the only case that we will be dealing with. We explain why in section 3.

In Section 4, we compute the  $BV_n$ -algebra structure on  $H_*(\Omega^n X; \mathbb{Q})$  when  $X$  is  $n$ -connected: we show that the  $BV_n$ -algebra  $H_*(\Omega^n X; \mathbb{Q})$  is the  $BV_n$ -algebra *freely generated* by the Lie algebra  $\pi_*(\Omega X) \otimes \mathbb{Q}$  (Theorem 4.4).

In Section 5, we explain (Theorem 5.10) how the action of  $H_*SO(n)$  on  $H_*\Omega^n X$  is related to the  $J$ -homomorphism, a useful fact used later in our computations. In Section 4 (Theorem 4.4) over  $\mathbb{Q}$  and then more generally in Section 5 (Corollary 5.11 ii)) over any field  $\mathbb{K}$  of characteristic different from 2, we see that the  $BV$  operator vanishes on spherical classes.

In Section 6, we show (Theorem 6.2) that the  $BV$  operator

$$BV : H_1(\Omega^2 S^3, \mathbb{F}_2) \xrightarrow{\cong} H_2(\Omega^2 S^3, \mathbb{F}_2)$$

is an isomorphism. Therefore over  $\mathbb{F}_2$ , the  $BV$  operator is in general non-trivial on spherical classes.

Finally we notice that:

-Theorem 6.2 is a crucial step in the main result of [8] and

-Kallel and Salvatore [6, Proposition 7.46] have given us an independent proof of this theorem.

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## 2. $H_*(f\mathcal{D}_n; \mathbb{Q})$ -ALGEBRA STRUCTURES ON $H_*(\Omega^n X; \mathbb{Q})$

Denote by  $\Omega_k^n S^n$  the path-connected component of  $\Omega^n S^n$  given by pointed maps of degree  $k$ . For  $n = \infty$ , we denote the colimit of the spaces  $\Omega^n S^n$  under the suspension maps by  $QS^0$ , which is the infinite loop space associated to the stable homotopy groups of spheres, that is  $\pi_* QS^0$  is the ring  $\pi_*^S$  of stable homotopy. The degree  $i$  component of  $QS^0$  is denoted by  $Q_i S^0$ . In all cases,  $[k]$  is the element in  $\pi_0(\Omega^n S^n) \cong \mathbb{Z}$  represented by  $\Omega_k^n S^n$ . The special orthogonal group  $SO(n)$  acts on the sphere  $S^{n-1}$  in a non-pointed way. By considering the sphere  $S^n$  as the non-reduced suspension of  $S^{n-1}$ , we obtain a pointed  $SO(n)$ -action on  $S^n$ . For example, by rotating, the ‘earth’  $S^2$  has an action of the circle  $S^1$  preserving the North pole. By adjunction, we have a morphism of monoids  $\Theta : (SO(n), 1) \rightarrow (\Omega_1^n S^n, id_{S^n})$ . Recall that  $\Omega_1^n S^n$  is the monoid of self-homotopy equivalences homotopic to the identity.

**Proposition 2.1.** *Let  $X$  be a pointed  $SO(n)$ -space. Suppose that the action of  $SO(n)$  on  $X$  is obtained by restriction along the morphism  $\Theta : SO(n) \rightarrow \Omega_1^n S^n$  from some action of  $\Omega_1^n S^n$  on  $X$ . Then the action of  $H_i(SO(n); \mathbb{Q})$  on  $H_*(\Omega^n X; \mathbb{Q})$  coming from the diagonal action of  $SO(n)$  on  $\Omega^n X$  is trivial except possibly if  $n$  is even and  $i = n - 1$ .*

The case where  $n$  is even and  $i = n - 1$  is analyzed in Section 4. Here are the most important examples of pointed  $SO(n)$ -spaces. Note that we can apply Proposition 2.1 to them.

*Example 2.2.* Let  $Y$  be a pointed space. We consider the  $n$ th reduced suspension of  $Y$ , which is by definition  $X = S^n \wedge Y$ . The space  $X$  has a pointed  $SO(n)$ -action defined by  $g.(t \wedge y) := (g.t) \wedge y$  for any  $g \in SO(n)$ ,  $t \in S^n$  and  $y \in Y$ .

*Example 2.3.* Any pointed topological space  $X$  can be considered as a trivial pointed  $SO(n)$ -space.

We give immediately the proof of Proposition 2.1 since it will give us the opportunity to review in general the diagonal action of  $SO(n)$  on  $\Omega^n X$ , for a pointed  $SO(n)$ -space  $X$ .

*Proof.* Recall that an  $H$ -group is an associative  $H$ -space with an homotopy inverse. An  $H$ -map between two  $H$ -groups is called a *morphism of  $H$ -groups*. Let  $G$  be an  $H$ -group acting pointedly on two spaces  $Y$  and  $Z$ . The diagonal action on the space of pointed maps,  $map_*(Y, Z)$ , is the action defined by

$$(g.f)(y) := g.f(g^{-1}.y)$$

for any  $g \in G$ ,  $y \in Y$  and any pointed map  $f : (Y, *) \rightarrow (Z, *)$ . The diagonal action is natural up to homotopy with respect to morphisms of  $H$ -groups, since morphisms of  $H$ -groups commute necessarily up to homotopy with the homotopy inverses. Let  $G := SO(n)$ ,  $Y := S^n$  and let  $Z$  be any pointed  $SO(n)$ -space  $X$ , we obtain the action considered by Getzler, Salvatore and Wahl [9, Example 2.5].

The monoid  $(\Omega_1^n S^n, id)$  is path-connected. So it is a  $H$ -group [12, X.2.2] and  $\Theta : (SO(n), 1) \rightarrow (\Omega_1^n S^n, id)$  is a morphism of  $H$ -groups.

Now suppose that the pointed action of  $SO(n)$  on  $X$  is obtained by restriction of an action of  $\Omega_1^n S^n$ . Then by naturality with respect to  $\Theta$ , the diagonal action of  $SO(n)$  on  $\Omega^n X$  is homotopic to the composite

$$SO(n) \times \Omega^n X \xrightarrow{\Theta \times \Omega^n X} \Omega_1^n S^n \times \Omega^n X \xrightarrow{\text{action} \times \Omega^n X} \Omega^n X.$$

where action is the diagonal action of  $\Omega_1^n S^n$  on  $\Omega^n X$ .

As we will see in more details below (beginning of Section 5), there is a pointed homotopy equivalence between  $(\Omega_0^n S^n, *)$  and  $(\Omega_1^n S^n, id)$ . But

$$\forall i \geq 1, \quad \pi_i(\Omega_0^n S^n, *) \otimes \mathbb{Q} \cong \pi_{i+n}(S^n, *) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q} & \text{if } n \text{ is even and } i = n - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore for  $n$  even,  $\Omega_1^n S^n$  is rationally homotopy equivalent to  $S^{n-1}$ . And for  $n$  odd,  $\Omega_1^n S^n$  is rationally contractible.  $\square$

### 3. DECOMPOSITION OF THE ACTION OF $SO(n)$ ON $\Omega^n X$ FOR A GENERAL $SO(n)$ -SPACE

The diagonal action of  $SO(n)$  on  $\Omega^n X$  is the combination of two different actions of  $\Omega^n X$ :

- (1) The action denoted by  $S$ , ‘on the source’, given by  $(g.f)(y) := f(g^{-1}.y)$  for any  $g \in SO(n)$ ,  $y \in S^n$  and any pointed map  $f : (S^n, *) \rightarrow (Z, *)$ .
- (2) The action denoted by  $T$ , ‘on the target’, given by  $(g.f)(y) := g.f(y)$  for any  $g \in SO(n)$ ,  $y \in S^n$  and any pointed map  $f : (S^n, *) \rightarrow (Z, *)$ .

For  $n = 2$ , the result in this section is

**Theorem 3.1.** *Let  $X$  be a pointed  $S^1$ -space. Denote by  $BV_{diag}$ ,  $BV_S$  and  $BV_T$  the  $BV$ -operators respectively due to the diagonal action, the action ‘on the source’ and the action ‘on the target’ of  $S^1$  on  $\Omega^2 X$ . Then  $BV_{diag} = BV_S + BV_T$  and  $BV_T$  is a derivation with respect to the Pontryagin product.*

The diagonal action is in homology the composite (where the roles of  $S$  and  $T$  could be permuted)

$$\begin{array}{c}
 H_*SO(n) \otimes H_*\Omega^n X \\
 \downarrow \Delta_{H_*SO(n) \otimes H_*\Omega^n X} \\
 H_*SO(n) \otimes H_*SO(n) \otimes H_*\Omega^n X \\
 \downarrow H_*SO(n) \otimes H_*S \\
 H_*SO(n) \otimes H_*\Omega^n X \\
 \downarrow H_*T \\
 H_*\Omega^n X.
 \end{array}$$

For example when  $n = 2$ , let  $[S^1]$  be the fundamental class of  $H_*SO(2)$ . Since  $[S^1]$  is a primitive element, the  $BV$ -operator  $BV_{diag}$  on  $H_*\Omega^2 X$ , which is due to the diagonal action in homology of  $[S^1]$ , is the sum of the two operators  $BV_S$  and  $BV_T$  on  $H_*\Omega^2 X$  given by the action  $S$  on the source and the action  $T$  on the target of  $[S^1]$ .

Let  $\circ : map_*(X, Y) \times \Omega^n X \rightarrow \Omega^n Y$ ,  $(g, f) \mapsto g \circ f$  be the composition map. We have the following obvious distributive law between composition  $\circ$  and loop multiplication due the structure of co-H-group on  $S^n$ . For any  $f \in map_*(X, Y)$ ,  $g, h \in \Omega^n X$ ,

$$(3.2) \quad f \circ (gh) = (f \circ g)(f \circ h).$$

In the particular case  $Y = X$ , this means that the multiplication of loops

$$\Omega^n X \times \Omega^n X \rightarrow \Omega^n X$$

is  $map_*(X, X)$ -equivariant. Since  $X$  is a pointed  $SO(n)$ -space, we have a morphism of monoid  $SO(n) \rightarrow map_*(X, X)$ . Therefore the multiplication of loops

$$\Omega^n X \times \Omega^n X \rightarrow \Omega^n X$$

is  $SO(n)$ -equivariant with respect to the action ‘on the target’. So in homology, the Pontryagin product

$$H_*\Omega^n X \times H_*\Omega^n X \rightarrow H_*\Omega^n X$$

is a morphism of  $H_*SO(n)$ -modules for the action ‘on the target’.

For example, when  $n = 2$ , the action of  $[S^1]$  in homology ‘on the target’ gives an operator  $BV_T$  which is a derivation with respect to the Pontryagin product and so does not contribute at all to the bracket (See formula (1.3)). Therefore we do not find the action ‘on the target’ of  $SO(n)$  on  $\Omega^n X$  interesting. In the rest of the paper, we will study only the action ‘on the source’ of  $SO(n)$  on  $\Omega^n X$  or equivalently we will consider the diagonal action of  $SO(n)$  on  $\Omega^n X$  where  $X$  is a trivial pointed  $SO(n)$ -space (Example 2.3).

4. COMPUTATION OF THE  $H_*(f\mathcal{D}_n; \mathbb{Q})$ -ALGEBRA  $H_*(\Omega^n X; \mathbb{Q})$ 

**Freely generated  $e_n$ -algebras.** Let  $n \in \mathbb{Z}$ . The suspension of a graded vector space  $V$  is the graded vector space  $sV$  such that  $(sV)_{i+1} = V_i$ . Let  $L$  be a graded Lie algebra. The  $(n-1)$  desuspension of  $L$ ,  $s^{1-n}L$ , has a Lie bracket of degree  $n-1$ . The free graded commutative algebra  $\Lambda(s^{1-n}L)$  is equipped with a unique structure of  $e_n$ -algebra such that the inclusion  $s^{1-n}L \hookrightarrow \Lambda(s^{1-n}L)$  commutes with the brackets. This inclusion is universal. When  $n = 1$ , this bracket is the well-known Schouten bracket of the Poisson algebra  $\Lambda L$ . Let  $A$  be an  $e_n$ -algebra. Then any Lie algebra morphism  $s^{1-n}L \rightarrow A$  extends to a unique morphism of  $e_n$ -algebras  $\Lambda(s^{1-n}L) \rightarrow A$ . We will call this  $e_n$ -algebra  $\Lambda(s^{1-n}L)$ , the  $e_n$ -algebra freely generated by the graded Lie algebra  $L$ . Let  $L := \pi_*(\Omega X) \otimes \mathbb{Q}$  be equipped with the Samelson bracket and  $A := H_*(\Omega^n X; \mathbb{Q})$ . Then  $s^{1-n}L = \pi_*\Omega^n X \otimes \mathbb{Q}$  and using Milnor-Moore and Cartan-Serre theorems [3, Theorems 16.10 and 21.5] we have

**Theorem 4.1.** [2, Remark 1.2 p. 214][1] *Let  $n \geq 2$ . Let  $X$  be an  $n$ -connected topological space. The Hurewicz morphism induces an isomorphism*

$$\Lambda(\pi_*\Omega^n X \otimes \mathbb{Q}) \xrightarrow{\cong} H_*(\Omega^n X; \mathbb{Q})$$

of both  $e_n$ -algebras and Hopf algebras between the  $e_n$ -algebra freely generated by the Lie algebra  $\pi_*(\Omega X) \otimes \mathbb{Q}$  and the rational homology of the iterated loop space.

*Remark 4.2.* We notice that the tensor product of two  $e_n$ -algebras is naturally an  $e_n$ -algebra, where the bracket on tensors is given by:

$$\{a \otimes b, a' \otimes b'\} = (-1)^{|b|(|a'|+n-1)}\{a, a'\} \otimes bb' + (-1)^{|a'|(|b|+n-1)}aa' \otimes \{b, b'\}.$$

An  $e_n$ -algebra  $L$  which is also a Hopf algebra is called a graded Hopf  $e_n$ -algebra if its diagonal map preserves Lie brackets (this structure is called *Poisson Hopf algebra* if  $n = 1$ ). Freely generated  $e_n$ -algebras are actually graded Hopf  $e_n$ -algebras and have the universal property in the relevant category. On the other hand,  $H_*(\Omega^n X; \mathbb{Q})$  is a Hopf  $e_n$ -algebra and the isomorphism in Theorem 4.1 above is in fact an isomorphism of graded Hopf  $e_n$ -algebras.

**Freely generated  $BV_n$ -algebras.** Suppose now that  $n \in \mathbb{Z}$  is even and that the graded Lie algebra  $L$  is equipped with a differential  $d_L$ . Let  $d_0$  be the derivation of degree  $n-1$  on  $\Lambda(s^{1-n}L)$  given by

$$d_0(s^{1-n}x_1 \wedge \cdots \wedge s^{1-n}x_k) = - \sum_{i=1}^k (-1)^{n_i} s^{1-n}x_1 \wedge \cdots \wedge s^{1-n}d_L x_i \wedge \cdots \wedge s^{1-n}x_k$$

where  $n_i = \sum_{j < i} |s^{1-n}x_j|$ . Since  $d_L^2 = 0$ ,  $d_0^2 = 0$ . Let  $d_1$  be the endomorphism of degree  $n-1$  on  $\Lambda(s^{1-n}L)$  given by

$$d_1(s^{1-n}x_1 \wedge \cdots \wedge s^{1-n}x_k) = \sum_{1 \leq i < j \leq k} (-1)^{|x_i - n + 1|} (-1)^{n_{ij}} s^{1-n}\{x_i, x_j\} \wedge s^{1-n}x_1 \cdots \widehat{s^{1-n}x_i} \cdots \widehat{s^{1-n}x_j} \cdots \wedge s^{1-n}x_k.$$

The symbol  $\widehat{\phantom{x}}$  means ‘deleted’. Here the sign  $(-1)^{n_{ij}}$  is such that  $s^{1-n}x_1 \wedge \cdots \wedge s^{1-n}x_k = (-1)^{n_{ij}} s^{1-n}x_i \wedge s^{1-n}x_j \wedge s^{1-n}x_1 \wedge \cdots \widehat{s^{1-n}x_i} \cdots \widehat{s^{1-n}x_j} \cdots \wedge s^{1-n}x_k$ . Since  $d_L$  is a derivation with respect to the bracket, we have  $d_0 d_1 + d_1 d_0 = 0$ . The Jacobi identity implies that  $d_1^2 = 0$ .

Consider the endomorphism  $BV$  of degree  $n - 1$  on  $\Lambda(s^{1-n}L)$  defined by  $BV := d_0 + d_1$ . We have  $BV \circ BV = 0$ . A direct calculation shows that

$$(-1)^{|a|} \left( BV(ab) - (BVa)b - (-1)^{|a|} a(BVb) \right)$$

is the bracket on the  $e_n$ -algebra  $\Lambda(s^{1-n}L)$ . Therefore the  $e_n$ -algebra  $\Lambda(s^{1-n}L)$  equipped with this linear operator  $BV$  is a  $BV_n$ -algebra, that we will denote  $\Lambda s^{1-n}(L, d_L)$  and that we will call the  *$BV_n$ -algebra freely generated by the differential graded Lie algebra  $(L, d_L)$* . The inclusion  $s^{1-n}(L, d_L) \hookrightarrow \Lambda s^{1-n}(L, d_L)$  commutes with the brackets and the differentials  $-d_L$  and  $BV$ . Again, this inclusion is universal. Let  $A$  be a  $BV_n$ -algebra. Then any differential graded Lie algebra morphism  $s^{1-n}(L, d_L) \rightarrow A$  extends to a unique morphism of  $BV_n$ -algebras  $\Lambda s^{1-n}(L, d_L) \rightarrow A$ . To sum up, the construction of the freely generated  $BV_n$ -algebra is the left adjoint functor to the forgetful functor from  $BV_n$ -algebras to differential graded Lie algebras.

Suppose moreover that  $\mathbb{K} := \mathbb{Q}$ . Then the differential  $d_1$  is the unique coderivation on  $\Lambda(s^{1-n}L)$  decreasing wordlength by 1 such that

$$(4.3) \quad d_1(s^{1-n}x \wedge s^{1-n}y) = (-1)^{|x|-n+1} s^{1-n}\{x, y\}.$$

When  $n = 0$ , our operator  $BV$  on  $\Lambda(s^{1-n}L)$  coincides with the differential of the Cartan-Chevalley-Eilenberg complex [3, p. 301] whose homology is  $\mathrm{Tor}_*^{U(L, d_L)}(\mathbb{Q}, \mathbb{Q})$ . The case  $n = 2$  is considered in [11, Section 1.1 p. 10]. We prove

**Theorem 4.4.** *Suppose that  $n \geq 2$  and  $n$  is even. Let  $X$  be an  $n$ -connected topological space. The  $BV_n$ -algebra  $H_*(\Omega^n X; \mathbb{Q})$  is isomorphic to  $\Lambda s^{1-n}(\pi_* \Omega X \otimes \mathbb{Q}, 0)$ , the  $BV_n$ -algebra freely generated by the graded Lie algebra  $\pi_* \Omega X \otimes \mathbb{Q}$  equipped with the zero differential.*

*Remark 4.5.* The tensor product of two  $BV_n$ -algebras is naturally a  $BV_n$ -algebra. The  $BV$  operator on tensors is given by

$$BV(a \otimes b) = BV(a) \otimes b + (-1)^{|a|} a \otimes BV(b).$$

A  $BV_n$ -algebra which is also a Hopf algebra is called a Hopf  $BV_n$ -algebra if its diagonal is a map of  $BV_n$ -algebras. One can check that freely generated  $BV_n$ -algebras are actually Hopf  $BV_n$ -algebras, and are also universal for this structure. On the other hand, in  $H_*(\Omega^n X; \mathbb{Q})$  is actually a Hopf  $BV_n$ -algebra, and the isomorphism of Theorem 4.4 is in fact an isomorphism of Hopf  $BV_n$ -algebras. This is similar to Remark 4.2.

*Proof.* By Theorem 4.1, the underlying  $e_n$ -algebras are isomorphic. By the universal property of the freely generated  $BV_n$ -algebra  $\Lambda s^{1-n}(L, 0)$ , it suffices to show that the  $BV$ -operator on  $H_*(\Omega^n X; \mathbb{Q})$  vanishes on spherical classes.

The  $BV$ -operator on  $H_*(\Omega^n X; \mathbb{Q})$  is induced by the action of a primitive element of degree  $n - 1$  in  $H_*(SO(n); \mathbb{Q})$ . More generally, let  $Y$  be an  $SO(n)$ -space. The operator induced by the action of a primitive element in  $H_*SO(n)$  is a coderivation (coaugmented if the action is pointed). The image of a primitive element by a coaugmented coderivation is a primitive element. Therefore, the operator  $BV$  induces an operator of degree  $n - 1$  on the primitive elements of  $H_*(\Omega^n X; \mathbb{Q})$ , denoted by  $PH_*(\Omega^n X; \mathbb{Q})$ . Let  $i \geq 1$ . By Cartan-Serre's theorem,  $PH_i(\Omega^n X; \mathbb{Q}) \cong \pi_i \Omega^n X \otimes \mathbb{Q} \cong \pi_{i+n} X \otimes \mathbb{Q}$ . Denote by  $X_{\mathbb{Q}}$  the rationalization of  $X$ . By the universal property of localization,  $\pi_{i+n} X \otimes \mathbb{Q} \cong \pi_{i+n} X_{\mathbb{Q}} \cong [S_{\mathbb{Q}}^{i+n}, X_{\mathbb{Q}}]$ . So

the operator  $BV$  can be identified with a morphism between the pointed homotopy classes

$$[S_{\mathbb{Q}}^{i+n}, X_{\mathbb{Q}}] \rightarrow [S_{\mathbb{Q}}^{i+2n-1}, X_{\mathbb{Q}}].$$

Now since  $X$  was equipped with the structure of trivial  $SO(n)$ -pointed space, the  $BV$  operator is natural with respect to based continuous maps. This  $BV$  operator can be therefore identified with a ‘rational homotopy operation’: a natural transformation  $[S_{\mathbb{Q}}^{i+n}, X_{\mathbb{Q}}] \rightarrow [S_{\mathbb{Q}}^{i+2n-1}, X_{\mathbb{Q}}]$ . By [12, XI.1.2], this rational homotopy operation is the composition by an element of  $\pi_{i+2n-1}S^{i+n} \otimes \mathbb{Q}$ . Since for  $i \geq 1$ ,  $\pi_{i+2n-1}S^{i+n} \otimes \mathbb{Q}$  is trivial, the  $BV$  operator is zero on  $\pi_i\Omega^n X \otimes \mathbb{Q}$  for  $i \geq 1$ .  $\square$

Remark that in fact, in this proof, we have shown that the  $BV$ -operator on  $H_*(\Omega^n X; \mathbb{Q})$  is the unique coderivation decreasing wordlength by 1 satisfying (4.3). Therefore we have recovered without computations that  $\Lambda(s^{-(n-1)}L)$  is a  $BV_n$ -algebra in the case  $L := \pi_*\Omega X \otimes \mathbb{Q}$ . This was our starting observation.

*Remark 4.6.* Suppose more generally that  $X$  is an  $SO(n)$ -pointed space not necessarily trivial. The generator in degree  $n - 1$  which defines the structure of  $BV_n$ -algebra (Theorem 1.5) on  $H_*(\Omega^n X; \mathbb{Q})$  is also primitive. Therefore Theorem 3.1 holds also for the operator  $BV = BV_{diag}$  of degree  $n - 1$  on  $H_*(\Omega^n X; \mathbb{Q})$ . In particular,  $BV = BV_S + BV_T$ . Moreover, this operator  $BV$  induces a differential  $d_L$  on  $\pi_*\Omega X \otimes \mathbb{Q}$ . By (1.4) and [2, Remark 1.2 p. 214],  $d_L$  is a derivation with respect to the Samelson bracket. And this time, the  $BV_n$ -algebra  $H_*(\Omega^n X; \mathbb{Q})$  is isomorphic to  $\Lambda^{s^{1-n}}(\pi_*\Omega X \otimes \mathbb{Q}, d_L)$  the freely generated  $BV_n$ -algebra with the (in general non-zero) differential  $d_L$ . By Theorem 4.4, the differential  $d_1$  corresponds to  $BV_S$  while  $d_0$  corresponds to  $BV_T$ .

Finally, remark that if  $X = M$  is a manifold and  $n = 2$ , Salvatore and Wahl [9, Theorem 6.5] have also given a formula for the  $BV_2$ -algebra  $H_*(\Omega^2 M; \mathbb{Q})$ .

In the following section, we show that over any field  $\mathbb{K}$ , the action of a spherical class in  $H_*SO(n)$  on a spherical class in  $H_*\Omega^n X$  still corresponds to a homotopy operation. But this homotopy operation is generally not trivial and is related to the well-known  $J$ -homomorphism.

## 5. ACTION OF $SO(n)$ ON ITERATED LOOP SPACES AND $J$ -HOMOMORPHISM

Let  $n \geq 2$ . The case of infinite loop spaces is implicitly included as the case  $n = \infty$ . In this case,  $SO(n)$  is meant to be the infinite orthogonal group  $SO$ . The iterated loop space  $(\Omega^n S^n, *)$  is an H-group. So the multiplication by  $id$  gives a free homotopy equivalence

$$\text{multiplication by } id : \Omega_0^n S^n \xrightarrow{\simeq} \Omega_1^n S^n.$$

Since both  $(\Omega_0^n S^n, *)$  and  $(\Omega_1^n S^n, id)$  are connected H-spaces, pointed homotopy classes coincide with free homotopy classes [12, III.1.11 and III.4.18]. So a based map  $(\Omega_0^n S^n, *) \rightarrow (\Omega_1^n S^n, id)$  freely homotopic to the multiplication by  $id$ , will give a pointed homotopy equivalence from  $(\Omega_0^n S^n, *)$  to  $(\Omega_1^n S^n, id)$ . The pointed homotopy inverse of this map is a map homotopic to the multiplication by a degree  $-1$  map from  $S^n$  to  $S^n$ . Let  $inv : SO(n) \rightarrow SO(n)$  be the map sending an orthogonal matrix to its inverse. Recall that we have a morphism of topological monoids

$$\Theta : (SO(n), 1) \rightarrow (\Omega_1^n S^n, id) .$$

Denote by  $ad_n : \pi_{i+n}(X) \xrightarrow{\cong} \pi_i(\Omega^n X)$  the adjunction map. The classical  $J$ -homomorphism that we will denote  $\mathbb{J}$ , is the composite (Compare with Madsen-Milgram p. 47, Igusa p. 247, Kono-Tamaki p. 113. or better John Klein's answer to the question "Fibrewise homotopy-equivalence of unit sphere bundles vs isomorphism of tangent bundles" in mathoverflow)

$$\pi_i(SO(n), 1) \xrightarrow{\pi_i(J)} \pi_i(\Omega^n S^n, *) \xrightarrow{ad_n^{-1}} \pi_{i+n}(S^n, *)$$

where  $J$  is the composite

$$(5.1) \quad J : (SO(n), 1) \xrightarrow{\Theta} (\Omega_1^n S^n, id) \xleftarrow{\cong} (\Omega_0^n S^n, *) .$$

Let  $J'$  be the map

$$J' : (SO(n), 1) \xrightarrow{inv} (SO(n), 1) \xrightarrow{\Theta} (\Omega_1^n S^n, id) \xleftarrow{\cong} (\Omega_0^n S^n, *)$$

That is  $J'$  is the precomposition of  $J$  with the inverse map of  $SO(n)$ . In homotopy,  $inv$  gives the inverse map. In homology,  $inv$  gives the antipode  $\chi$  of the Hopf algebra  $H_*SO(n)$ . Therefore, we have

**Lemma 5.2.** *In homotopy,  $\pi_* J' = -\pi_* J$  while in homology  $J'_* = J_* \chi$  where  $\chi$  is the antipode of the Hopf algebra  $H_*SO(n)$ . In particular, on a primitive element  $a \in H_*SO(n)$ ,  $J'_*(a) = -J_*(a)$ .*

Let  $\circ : (\Omega^n X, *) \times (\Omega^n S^n, *) \rightarrow (\Omega^n X, *)$ ,  $(g, f) \mapsto g \circ f$  be the composition map. Using the distributive law (3.2), we have for any  $g \in \Omega^n X$ ,  $f \in \Omega^n S^n$ :

$$(5.3) \quad g \circ (f \cdot id_{S^n}) = (g \circ f)(g \circ id_{S^n}) = (g \circ f)g .$$

The left action of  $SO(n)$  on  $\Omega^n X$  is given by the diagram commutative up to free homotopy

$$\begin{array}{ccc} SO(n) \times \Omega^n X & \xrightarrow{J' \times \Omega^n X} & \Omega_0^n S^n \times \Omega^n X \\ (\Theta \circ inv) \times \Omega^n X \downarrow & \swarrow \text{mult by id} & \downarrow \Omega^n S^n \times \Delta_{\Omega^n X} \\ \Omega_1^n S^n \times \Omega^n X & & \Omega_0^n S^n \times \Omega^n X \times \Omega^n X \\ \tau \downarrow & & \downarrow \tau \times \Omega^n X \\ \Omega^n X \times \Omega_1^n S^n & & \Omega^n X \times \Omega_0^n S^n \times \Omega^n X \\ \circ \downarrow & & \downarrow \circ \times \Omega^n X \\ \Omega^n X & \xleftarrow{\text{mult}} & \Omega^n X \times \Omega^n X \end{array}$$

Here  $\tau : \Omega^n S^n \times \Omega^n X \rightarrow \Omega^n X \times \Omega^n S^n$  is the map that interchanges factors. The top left triangle commute up to free homotopy, by definition of  $J'$ . The bottom diagram commutes exactly by formula (5.3). Denote by  $\circ : H_*\Omega^n X \otimes H_*\Omega^n S^n \rightarrow H_*\Omega^n X$ ,  $(f \otimes g) \mapsto f \circ g$ , the morphism induced in homology by composition and by  $J_* : H_*SO(n) \rightarrow H_*\Omega^n S^n$ , the morphism induced by  $J$ .

**Proposition 5.4.** *In homology, the action of  $H_*SO(n)$  on  $H_*\Omega^n X$ , denoted by  $\cdot$ , is given for  $f \in H_*SO(n)$  and  $g \in H_*\Omega^n X$  by*

$$f \cdot g = \sum_i (-1)^{|g'_i||f|} (g'_i \circ (J'_* f)) g''_i$$

where  $\Delta g = \sum_i g'_i \otimes g''_i$ .

Note that alternatively to give a shorter proof of this proposition, we could have applied [2, Theorem 3.2 i) p. 363]. But we have preferred to give a independent simple proof.

The unit 1 of the algebra  $H_*\Omega^n X$  is given by applying homology to the inclusion of the constant map  $*$   $\hookrightarrow$   $\Omega^n X$ . Therefore by applying homology to the commuting diagram

$$\begin{array}{ccc} * \times \Omega^n S^n & \longrightarrow & * \\ \downarrow & & \downarrow \\ \Omega^n X \times \Omega^n S^n & \xrightarrow{\circ} & \Omega^n X \end{array}$$

we obtain [2, Proposition 3.7(i) p. 364] that for any  $f \in H_*\Omega^n S^n$ ,

$$1 \circ f = \varepsilon(f)1.$$

Here  $\varepsilon$  is the augmentation of the Hopf algebra  $H_*\Omega^n S^n$ . So, in homology, the action of  $f \in H_{>0}SO(n)$  on a primitive element  $g \in H_*\Omega^n X$  is given by

$$(5.5) \quad f \cdot g = (-1)^{|f||g|} g \circ (J_* f) = (-1)^{|f||g|+1} g \circ (J_* f) \quad .$$

Which means that the action of  $H_{>0}SO(n)$  on primitive elements of  $H_*\Omega^n X$  is given in homology by the composite

$$(SO(n), 1) \times (\Omega^n X, *) \xrightarrow{J' \times \Omega^n X} (\Omega^n S^n, *) \times (\Omega^n X, *) \xrightarrow{\tau} (\Omega^n X, *) \times (\Omega^n S^n, *) \xrightarrow{\circ} (\Omega^n X, *).$$

Let  $\bar{\circ} : \pi_i(\Omega^n X, *) \times \pi_j(\Omega^n S^n, *) \rightarrow \pi_{i+j}(\Omega^n X, *)$  be the map induced on homotopy groups by the composition  $\circ$ . Explicitly, the composition map

$$\circ : (\Omega^n X, *) \times (\Omega^n S^n, *) \rightarrow (\Omega^n X, *)$$

passing to the quotient, defines a map

$$(\Omega^n X, *) \wedge (\Omega^n S^n, *) \rightarrow (\Omega^n X, *).$$

The map  $\bar{\circ}$  sends  $(f, g)$  to the composite

$$f \bar{\circ} g : S^i \wedge S^j \xrightarrow{f \wedge g} (\Omega^n X, *) \wedge (\Omega^n S^n, *) \rightarrow (\Omega^n X, *).$$

Let  $X$  be an infinite loop space, the  $\bar{\circ}$  product extends to a product

$$\bar{\circ} : (X, *) \times (QS^0, *) \rightarrow (X, *) \quad .$$

As before, the  $\circ$  product induces a map

$$\bar{\circ} : \pi_i X \times \pi_j QS^0 \longrightarrow \pi_{i+j} X \quad .$$

Denote by  $hur$  the Hurewicz morphism. Since the diagram

$$\begin{array}{ccc} S^i \times S^j & \xrightarrow{f \times g} & \Omega^n X \times \Omega^n S^n \\ \downarrow & & \downarrow \circ \\ S^i \wedge S^j & \xrightarrow{f \bar{\circ} g} & \Omega^n X \end{array}$$

is commutative, by applying homology we obtain that the Hurewicz morphism commutes with the composition:

$$(5.6) \quad hur(f) \circ hur(g) = hur(f \bar{\circ} g).$$

This proof is similar to the proof that  $hur$  commutes with the Samelson bracket [12, X.6.3]. From formulas (5.5) and (5.6), we deduce

**Proposition 5.7.** *Let  $f \in \pi_i SO(n)$ ,  $g \in \pi_j \Omega^n X$ ,  $i, j \geq 1$ . Then the action of  $hur(f) \in H_i SO(n)$  on  $hur(g) \in H_j \Omega^n X$  is given by*

$$hur(f) \cdot hur(g) = -(-1)^{ij} hur(g \bar{\circ} \pi_* J(f)) = -(-1)^{ij} hur(g \bar{\circ} ad_n(\mathbb{J}f)).$$

Recall that  $ad_n : \pi_{i+n} X \xrightarrow{\cong} \pi_i \Omega^n X$  denotes the adjunction map. The following lemma follows from elementary manipulation with the loop-suspension adjunction.

**Lemma 5.8.** *We have the following commutative diagram:*

$$\begin{array}{ccc} \pi_{i+n} X \times \pi_{j+n} S^n & \longrightarrow & \pi_{i+j+n} X \\ ad_n \times ad_n \downarrow \cong & & \cong \downarrow ad_n \\ \pi_i \Omega^n X \times \pi_j \Omega^n S^n & \xrightarrow{\bar{\circ}} & \pi_{i+j} \Omega^n X \end{array}$$

where the top arrow is the map sending  $(f, g)$  to  $f \circ \Sigma^i g$ , the composite of  $f$  and the  $i$ th suspension of  $g$ .

In the stable case, we simply obtain

**Lemma 5.9.** *Let  $X$  be an infinite loop space, and let  $\widehat{X}$  be the corresponding spectrum. The  $\bar{\circ}$  product coincides under the isomorphism  $\pi_* X \cong \pi_*^S \widehat{X}$  with the natural action of the stable homotopy ring  $\pi_*^S$  on the stable homotopy module  $\pi_*^S \widehat{X}$ .*

Recall that  $\mathbb{J} : \pi_i SO(n) \rightarrow \pi_{i+n}(S^n)$  denote the classical  $J$ -homomorphism.

**Theorem 5.10.** *Let  $f \in \pi_i SO(n)$ ,  $g \in \pi_{j+n} X$ ,  $i, j \geq 1$ . Then the action of  $hur(f) \in H_i SO(n)$  on  $(hur \circ ad_n)(g) \in H_j \Omega^n X$  is given by*

$$hur(f) \cdot (hur \circ ad_n)(g) = -(-1)^{ij} (hur \circ ad_n)(g \circ \Sigma^j \mathbb{J}f).$$

*Proof.* By applying Proposition 5.7 to  $f$  and  $ad_n g$ , and then Lemma 5.8 (or 5.9 if  $n = \infty$ )

$$hur(f) \cdot hur(ad_n g) = -(-1)^{ij} hur((ad_n g) \bar{\circ} ad_n(\mathbb{J}f)) = -(-1)^{ij} hur \circ ad_n(g \circ \Sigma^j \mathbb{J}f).$$

□

**Corollary 5.11.** *Let  $X$  be a topological space. Let  $g \in \pi_{j+2}(X)$ ,  $j \geq 1$ .*

*i) Consider the BV operator  $H_j(\Omega^2 X) \rightarrow H_{j+1}(\Omega^2 X)$ . Then*

$$BV((hur \circ ad_2)(g)) = -(-1)^j (hur \circ ad_2)(g \circ \Sigma^j \eta).$$

*ii) In particular, over a field  $\mathbb{K}$  of characteristic different from 2. the BV operator  $H_j \Omega^2 X \rightarrow H_{j+1} \Omega^2 X$  is zero on spherical classes.*

*Proof.* Take  $n = 2$  and let  $f := id_{S^1}$  in Theorem 5.10. The Hopf map  $\eta : S^3 \rightarrow S^2$  is equal to  $\mathbb{J}(id_{S^1})$ . Its  $j$ th suspension,  $\Sigma^j \eta$  is the generator of  $\pi_{3+j} S^{2+j} \cong \mathbb{Z}/2\mathbb{Z}$ . Therefore localized away from 2,  $\Sigma^j \eta$  is null homotopic. □

Recall that an element in the homology of a space is called *spherical* if it sits in the image of the Hurewicz homomorphism.

6. BATALIN-VILKOVISKY STRUCTURE OF  $H_*(\Omega^2 S^3; \mathbb{F}_2)$ 

In this section, we see that part ii) of Corollary 5.11 is not true in characteristic 2. In this last section, homology will mean homology with coefficients in  $\mathbb{F}_2$  and for any pointed space  $X$ ,  $hur : \pi_* X \rightarrow H_*(X, \mathbb{F}_2)$  will denote the modulo 2 Hurewicz homomorphism. Recall the following classical splitting lemma:

**Lemma 6.1.** *Let  $F \xrightarrow{j} E \xrightarrow{p} B$  be a homotopy fibration sequence (in the sense of [10, p. 53]) with  $F$ ,  $E$  and  $B$  path-connected. Suppose that there is a map  $s : B \rightarrow E$  such that the composite  $p \circ s$  is a homotopy equivalence and that  $E$  is an  $H$ -space with multiplication  $\mu$ . Then the composite  $\mu \circ (j \times s) : F \times B \xrightarrow{\cong} E$  is a homotopy equivalence. In particular [10, Corollary 7.1.5]  $j$  admits a retract up to homotopy.*

The homology of  $\Omega^2 S^3$ , as a Pontryagin algebra, is polynomial on generators  $u_n$  of degree  $2^n - 1$ ,  $n \geq 1$ , (see Cohen's work in [2]). We first notice that  $u_1$  is the bottom non-trivial class in positive degrees, and as such, must be in the image of the Hurewicz homomorphism, according to the Hurewicz theorem. But  $\pi_1 \Omega^2 S^3$  is infinite cyclic generated by  $\iota = \text{ad}_2(\text{Id}_{S^3})$  the adjoint of the identity of  $S^3$ .

Since  $u_1 = hur \iota = hur \circ \text{ad}_2(\text{id}_{S^3})$ , thanks to part i) of Corollary 5.11,

$$BV(u_1) = hur(\text{ad}_2(\Sigma\eta))$$

where  $\text{ad}_2(\Sigma\eta) \in \pi_2(\Omega^2 S^3)$  is the adjoint of suspension of the Hopf map  $\Sigma\eta \in \pi_4 S^3$ .

We want to show that the Hurewicz homomorphism for  $\Omega^2 S^3$  is non-trivial in degree 2. This can be seen as follows. We know that  $\pi_1 \Omega^2 S^3 \cong \mathbb{Z}$  and  $\pi_2 \Omega^2 S^3 \cong \mathbb{Z}/2\mathbb{Z}$  generated by  $\text{ad}_2(\Sigma\eta)$ .

Let  $p : S^3 \rightarrow K(\mathbb{Z}, 3)$  represent the generator for the third integral cohomology group of  $S^3$ ,  $\pi_3 K(\mathbb{Z}, 3) \cong H^3(S^3; \mathbb{Z})$ . Let  $S^3\langle 3 \rangle$  be the homotopy fiber of  $p$  and let  $j : S^3\langle 3 \rangle \rightarrow S^3$  be the fiber inclusion.

Let  $\iota : S^1 \rightarrow \Omega^2 S^3$  the adjoint of the identity of  $S^3$ . Since  $S^3\langle 3 \rangle$  is 3-connected,  $\pi_1(\Omega^2 p)$  is an isomorphism and so maps  $\iota$  to  $\pm \text{id}_{S^1}$ . Therefore by applying Lemma 6.1 to the homotopy fibration sequence  $\Omega^2 S^3\langle 3 \rangle \xrightarrow{\Omega^2 j} \Omega^2 S^3 \xrightarrow{\Omega^2 p} \Omega^2 K(\mathbb{Z}, 3) \simeq S^1$ , we obtain that  $\Omega^2 j : \Omega^2 S^3\langle 3 \rangle \rightarrow \Omega^2 S^3$  has a retract up to homotopy and so is injective in homology. Since  $\pi_3(S^1) = \pi_2(S^1) = 0$ ,  $\pi_2(\Omega^2 j) : \pi_2(\Omega^2 S^3\langle 3 \rangle) \xrightarrow{\cong} \pi_2(\Omega^2 S^3) \cong \mathbb{Z}/2\mathbb{Z}$  is an isomorphism. Since  $\Omega^2 S^3\langle 3 \rangle$  is simply connected, the Hurewicz homomorphism  $hur : \pi_2(\Omega^2 S^3\langle 3 \rangle) \xrightarrow{\cong} H_2(\Omega^2 S^3\langle 3 \rangle)$  is an isomorphism. Since  $H_2(\Omega^2 j)$  is also an isomorphism, the Hurewicz homomorphism

$$hur : \pi_2(\Omega^2 S^3) = \mathbb{Z}/2\mathbb{Z} \cdot \text{ad}_2(\Sigma\eta) \xrightarrow{\cong} H_2(\Omega^2 S^3) = \mathbb{Z}/2\mathbb{Z} \cdot u_1^2$$

is an isomorphism. Hence  $hur(\text{ad}_2(\Sigma\eta)) = u_1^2$  and so  $BV(u_1) = u_1^2$ . So we have proved

**Theorem 6.2.** *The Batalin-Vilkovisky operator  $BV : H_1(\Omega^2 S^3, \mathbb{F}_2) \xrightarrow{\cong} H_2(\Omega^2 S^3, \mathbb{F}_2)$  is non trivial.*

Note that for all  $n \geq 4$ , the Batalin-Vilkovisky operator  $BV : H_{n-2}(\Omega^2 S^n, \mathbb{F}_2) \rightarrow H_{n-1}(\Omega^2 S^n, \mathbb{F}_2) = \{0\}$  is obviously zero.

*Remark 6.3.* (to be compared with the last paragraph of Section 1 of [4, p. 271]) Since  $S^3$  is a double suspension,  $S^3$  is a  $S^1$ -pointed space (Example 2.2). Consider

the diagonal action on  $\Omega^2 S^3$ . The adjunction map  $X \rightarrow \Omega^2 \Sigma^2 X$  is  $S^1$ -equivariant with respect to the trivial action on  $X$  and the diagonal action on  $\Omega^2 \Sigma^2 X$ . So the  $BV$  operator due to the diagonal action

$$BV_{diag} : H_1(\Omega^2 S^3, \mathbb{F}_2) \rightarrow H_2(\Omega^2 S^3, \mathbb{F}_2)$$

is trivial. Therefore modulo 2, this operator  $BV_{diag}$  is different from the  $BV$  operator  $BV_S$  considered in Theorem 6.2. On the contrary, over  $\mathbb{Q}$ ,  $BV_{diag}$  and  $BV_S$  coincide on  $H_*(\Omega^2 \Sigma^2 X; \mathbb{Q})$  for any connected space  $X$ . Indeed, by Theorem 4.4 and (1.4), they both vanish on  $\pi_*(\Omega^2 \Sigma^2 X) \otimes \mathbb{Q}$  which is the desuspended free Lie algebra on  $H_{>0}(X)$ .

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GERALD GAUDENS, MATHEMATISCHES INSTITUT, BERINGSTRASSE 1, D-53115 BONN, GERMANY,

LUC MENICHI, UMR 6093 ASSOCIÉE AU CNRS, UNIVERSITÉ D'ANGERS, FACULTÉ DES SCIENCES, 2 BOULEVARD LAVOISIER, 49045 ANGERS, FRANCE

*E-mail address:* gaudens at math.uni-bonn.de, luc.menichi at univ-angers.fr