VAN DEN BERGH ISOMORPHISMS IN STRING TOPOLOGY

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ABSTRACT. Let M be a path-connected closed oriented d-dimensional smooth manifold and let $\mathbb{k}$ be a principal ideal domain. By Chas and Sullivan, the shifted free loop space homology of M, $H_{*+d}(LM)$ is a Batalin-Vilkovisky algebra. Let G be a topological group such that M is a classifying space of G. Denote by $S_*(G)$ the (normalized) singular chains on G. Suppose that G is discrete or path-connected. We show that there is a Van Den Bergh type isomorphism

$$HH^{-p}(S_*(G), S_*(G)) \cong HH_{p+d}(S_*(G), S_*(G)).$$

Therefore, the Gerstenhaber algebra $HH^*(S_*(G), S_*(G))$ is a Batalin-Vilkovisky algebra and we have a linear isomorphism

$$HH^*(S_*(G), S_*(G)) \cong H_{*+d}(LM).$$

This linear isomorphism is expected to be an isomorphism of Batalin-Vilkovisky algebras. We also give a new characterization of Batalin-Vilkovisky algebra in term of derived bracket.

1. INTRODUCTION

We work over an arbitrary principal ideal domain $\mathbb{k}$. Let M be a compact oriented d-dimensional smooth manifold. Denote by $LM := map(S^1, M)$ the free loop space on M. Chas and Sullivan [6] have shown that the shifted free loop homology $H_{*+d}(LM)$ has a structure of Batalin-Vilkovisky algebra (Definition 23). In particular, they showed that $H_{*+d}(LM)$ is a Gerstenhaber algebra (Definition 21). On the other hand, let A be a differential graded (unital associative) algebra. The Hochschild cohomology of A with coefficients in A, $HH^*(A, A)$, is a Gerstenhaber algebra. These two Gerstenhaber algebras are expected to be related:

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Conjecture 1. *Let G be a topological group such that M is a classifying space of G. There is an isomorphism of Gerstenhaber algebras $H_{*+d}(LM) \cong HH^*(S_*(G), S_*(G))$ between the free loop space homology and the Hochschild cohomology of the algebra of singular chains on G.*

Suppose that G is discrete or path-connected. In this paper, we define a Batalin-Vilkovisky algebra structure on $HH^*(S_*(G), S_*(G))$ and an isomorphism of graded $\Bbbk$-modules

$$BFG^{-1} \circ \mathcal{D} : H_{*+d}(LM) \cong HH^*(S_*(G), S_*(G))$$

which is compatible with the two Δ operators of the two Batalin-Vilkovisky algebras: $BFG^{-1} \circ \mathcal{D} \circ \Delta = \Delta \circ BFG^{-1} \circ \mathcal{D}$. Indeed, Burghelea, Fiedorowicz [5] and Goodwillie [19] gave an isomorphism of graded $\Bbbk$-modules

$$BFG : HH_*(S_*(G), S_*(G)) \overset{\cong}{\to} H_*(LM)$$

which interchanges Connes boundary map B and the Δ operator on $H_{*+d}(LM)$: $BFG \circ B = \Delta \circ BFG$. And in this paper, our main result is:

Theorem 2. *(Theorems 45 and 43) Let G be a discrete or a path-connected topological group such that its classifying space BG is an oriented Poincaré duality space of formal dimension d. Then*

a) there exists $\Bbbk$-linear isomorphisms

$$\mathcal{D} : HH_{d-p}(S_*(G), S_*(G)) \overset{\cong}{\to} HH^p(S_*(G), S_*(G)).$$

b) If B denotes Connes boundary map on $HH_*(S_*(G), S_*(G))$ then $\Delta := -\mathcal{D} \circ B \circ \mathcal{D}^{-1}$ defines a structure of Batalin-Vilkovisky algebra on $HH^*(S_*(G), S_*(G))$, extending the canonical Gerstenhaber algebra structure.

c) The cyclic homology of $S_*(G)$, $HC_*(S_*(G))$ has a Lie bracket of degre $2 - d$.

By [33, Proposition 28], c) follows directly from b). Note that when G is a discrete group, the algebra of normalized singular chains on G, $S_*(G)$ is just the group ring $\Bbbk[G]$.

To prove Conjecture 1 in the discrete or path-connected case, it suffices now to show that the composite $BFG^{-1} \circ \mathcal{D}$ is a morphism of graded algebras. When $\Bbbk$ is a field of characteristic 0 and G is discrete, this was proved by Vaintrob [40].

Suppose now that

(3) $\qquad M$ is simply-connected and that $\Bbbk$ is a field.

In this case, there is a more famous dual conjecture relating Hochschild cohomology and string topology.

Conjecture 4. *Under (3), there is an isomorphism of Gerstenhaber algebras* $H_{*+d}(LM) \cong HH^*(S^*(M), S^*(M))$ *between the free loop space homology and the Hochschild cohomology of the algebra of singular cochains on* M.

And in fact, Theorem 2 is the Eckmann-Hilton or Koszul dual of the following theorem.

Theorem 5. *(*[13, Theorem 23] *and* [33, Theorem 22]*) Assume (3).*
a) There exist isomorphism of graded $\mathbb{k}$*-vector spaces*

$$FTV : HH^{p-d}(S^*(M), S^*(M)^\vee) \xrightarrow{\cong} HH^p(S^*(M), S^*(M)).$$

b) The Connes coboundary $B^\vee$ *on* $HH^*(S^*(M), S^*(M)^\vee)$ *defines via the isomorphism* FTV *a structure of Batalin-Vilkovisky algebra extending the Gerstenhaber algebra* $HH^*(S^*(M), S^*(M))$.

Jones [23] proved that there is an isomorphism

$$J : H_{p+d}(LM) \xrightarrow{\cong} HH^{-p-d}(S^*(M), S^*(M)^\vee)$$

such that the Δ operator of the Batalin-Vilkovisky algebra $H_{*+d}(LM)$ and Connes coboundary map $B^\vee$ on $HH^{*-d}(S^*(M), S^*(M)^\vee)$ satisfies $J \circ \Delta = B^\vee \circ J$. Therefore, as we explain in [33], to prove conjecture 4, it suffices to show that the composite $FTV \circ J$ is a morphism of graded algebras.

In [12], together with Felix and Thomas, we prove that Hochschild cohomology satisfies some Eckmann-Hilton or Koszul duality.

Theorem 6. [12, Corollary 2]*(See also* [7, Theorem 69 and below]*) Let* $\mathbb{k}$ *be a field. Let* G *be a connected topological group. Denote by* $S^*(BG)$ *the algebra of singular cochains on the classifying space of* G. *Suppose that for all* $i \in \mathbb{N}$, $H_i(BG)$ *is finite dimensional. Then there exists an isomorphism of Gerstenhaber algebras*

$$Gerst : HH^*(S_*(G), S_*(G)) \xrightarrow{\cong} HH^*(S^*(BG), S^*(BG)).$$

Therefore under (3), Conjectures 4 and 1 are equivalent and under (3), Theorem 2 as stated in this introduction follows from Theorem 5.

The problem is that the isomorphism $Gerst$ in Theorem 6 does not admit a simple formula. On the contrary, as we explain in Theorems 45 and 43, in this paper, the isomorphism $\mathcal{D}$ is very simple: $\mathcal{D}^{-1}$ is given by the cap product with a fundamental class $c \in HH_d(S_*(G), S_*(G))$.

In [18, Theorem 3.4.3 i)], Ginzburg (See also [26, Proposition 1.4]) shows that for any Calabi-Yau algebra A, the Van den Bergh duality isomorphism $\mathcal{D} : HH_{d-p}(A,A) \overset{\cong}{\to} HH^p(A,A)$ is $HH^*(A,A)$-linear: $\mathcal{D}^{-1}$ is also given by the cap product with a fundamental class $c \in HH_d(A,A)$.

We now give the plan of the paper:

Section 2: We recall the definitions of the Bar construction, of the Hochschild (co)chain complex and of Hochschild (co)homology.

Section 3: We show that, for some augmented differential graded algebra A such that the dual of its reduced bar construction $B(A)^\vee$ satisfies Poincaré duality, we have a Van den Bergh duality isomorphism $HH_{d-p}(A,A) \cong HH^p(A,A)$ if A is connected (Corollaries 13 and 14).

Section 4: There is a well known isomorphism between group (co)homology and Hochschild (co)homology. We show that, through this isomorphism, cap products in Hochschild (co)homology correspond to cap products in group (co)homology.

Section 5: We give a new characterization of Batalin-Vilkovisky algebras.

Section 6: Ginzburg proved that if Hochschild (co)homology satisfies a Van den Bergh duality isomorphism $HH_{d-p}(A,A) \cong HH^p(A,A)$ then Hochschild cohomology has a Batalin-Vilkovisky algebra structure. We rewrite the proof of Ginzburg using our new characterization of Batalin-Vilkovisky algebras and insisting on signs.

Section 7: We show that a differential graded algebra quasi-isomorphic to an algebra satisfying Poincaré duality, also satisfies Poincaré duality (Proposition 41). Finally, we show our main theorem for path-connected topological group.

Section 8: We show our main theorem for discrete groups. Extending a result of Kontsevich [18, Corollary 6.1.4] and Lambre [26, Lemme 6.2], we also show that, over any commutative ring $\mathbb{k}$, the group ring $\mathbb{k}[G]$ of an orientable Poincaré duality group is a Calabi-Yau algebra.

Section 9: Let G be a path-connected compact Lie group of dimension d. We give another Van Den Bergh type isomorphism

$$HH^p(S^*(BG), S^*(BG)) \cong HH_{-d-p}(S^*(BG), S^*(BG)).$$

Therefore, the Gerstenhaber algebra $HH^*(S^*(BG), S^*(BG))$ is a Batalin-Vilkovisky algebra and we have a linear isomorphism

$$HH^*(S^*(BG), S^*(BG)) \cong H^{*+d}(LBG).$$

Appendix: We recall the notion of derived bracket following Kosmann-Schwarzbach [24]. We interpret our new characterization of Batalin-Vilkovisky algebra in term of derived bracket (Theorem 66). To any differential graded algebra A, we associate

-a new Lie bracket on A (Remark 63),

-a new Gerstenhaber algebra which is a sub algebra of the endomorphism algebra of $HH_*(A, A)$ (Theorem 67).

We conjecture that Theorem 2 is true without assuming that G is discrete or path-connected. Note that the proof of the discrete case (Sections 4 and 8) is independent of the proof of the path-connected case (Sections 3 and 7).

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2. Hochschild homology and cohomology

We work over an arbitrary commutative ring $\Bbbk$ except in sections 3 and 7, where $\Bbbk$ is assumed to be a principal ideal domain and in section 9 where $\Bbbk$ is assumed to be a field. We use the graded differential algebra of [11, Chapter 3]. In particular, an element of lower degree $i \in \mathbb{Z}$ is by the *classical convention* [11, p. 41-2] of upper degree $-i$. Differentials are of lower degree -1. All the algebras considered in this paper, are unital and associative. Let A be a differential graded algebra. Let M be a right A-module and N be a left A-module. Denote by sA the suspension of A, $(sA)_i = A_{i-1}$. Let d_0 be the differential on the tensor product of complexes $M \otimes T(sA) \otimes N$. We denote the tensor product of the elements $m \in M$, $sa_1 \in sA$, $\dots$, $sa_k \in sA$ and $n \in N$ by $m[a_1|\cdots|a_k]n$. Let d_1 be the differential on the graded vector space $M \otimes T(sA) \otimes N$ defined by:

$$\begin{aligned} d_1 m[a_1|\cdots|a_k]n =& (-1)^{|m|} m a_1[a_2|\cdots|a_k]n \\ &+ \sum_{i=1}^{k-1} (-1)^{\varepsilon_i} m[a_1|\cdots|a_i a_{i+1}|\cdots|a_k]n \\ &- (-1)^{\varepsilon_{k-1}} m[a_1|\cdots|a_{k-1}]a_k n; \end{aligned}$$

Here $\varepsilon_i = |m| + |a_1| + \cdots + |a_i| + i$.

The *bar construction of A with coefficients in M and in N*, denoted $B(M; A; N)$, is the complex $(M \otimes T(sA) \otimes N, d_0 + d_1)$. The *bar resolution of A*, denoted $B(A; A; A)$, is the differential graded (A, A)-bimodule $(A \otimes T(sA) \otimes A, d_0 + d_1)$. If A is augmented then the *reduced bar construction of A*, denoted $B(A)$, is $B(\Bbbk; A; \Bbbk)$.

Denote by A^{op} the opposite algebra of A and by $A^e := A \otimes A^{op}$ the envelopping algebra of A. Let M be a differential graded (A, A)-bimodule. Recall that any (A, A)-bimodule can be considered as a left (or right) A^e-module. The *Hochschild chain complex* is the complex $M \otimes_{A^e} B(A; A; A)$ denoted $\mathcal{C}_*(A, M)$. Explicitly $\mathcal{C}_*(A, M)$ is the complex $(M \otimes T(sA), d_0 + d_1)$ with d_0 obtained by tensorization and [8, (10) p. 78]

$$\begin{aligned} d_1 m[a_1|\cdots|a_k] =& (-1)^{|m|} m a_1[a_2|\cdots|a_k] \\ &+ \sum_{i=1}^{k-1} (-1)^{\varepsilon_i} m[a_1|\cdots|a_i a_{i+1}|\cdots|a_k] \\ &- (-1)^{|sa_k|\varepsilon_{k-1}} a_k m[a_1|\cdots|a_{k-1}]. \end{aligned}$$

The *Hochschild homology of A with coefficients in M* is the homology H of the Hochschild chain complex:

$$HH_*(A, M) := H(\mathcal{C}_*(A, M)).$$

The *Hochschild cochain complex* of A with coefficients in M is the complex $\mathrm{Hom}_{A^e}(B(A; A; A), M)$ denoted $\mathcal{C}^*(A, M)$. Explicitly $\mathcal{C}^*(A, M)$ is the complex

$$(\mathrm{Hom}(T(sA), M), D_0 + D_1).$$

Here for $f \in \mathrm{Hom}(T(sA), M)$, $D_0(f)([\,]) = d_M(f([\,]))$, $D_1(f)([\,]) = 0$, and for $k \geq 1$, we have:

$$D_0(f)([a_1|a_2|...|a_k]) = d_M(f\left([a_1|a_2|...|a_k]\right)) - \sum_{i=1}^{k} (-1)^{\overline{\epsilon}_i} f([a_1|...|d_A a_i|...|a_k])$$

and

$$\begin{aligned} D_1(f)([a_1|a_2|...|a_k]) = \quad & -(-1)^{|sa_1|\,|f|} a_1 f([a_2|...|a_k]) \\ & - \textstyle\sum_{i=2}^{k} (-1)^{\overline{\epsilon}_i} f([a_1|...|a_{i-1}a_i|...|a_k]) \\ & + (-1)^{\overline{\epsilon}_k} f([a_1|a_2|...|a_{k-1}]) a_k\,, \end{aligned}$$

where $\overline{\epsilon}_i = |f| + |sa_1| + |sa_2| + ... + |sa_{i-1}|$.

The *Hochschild cohomology of A with coefficients in M* is

$$HH^*(A, M) = H(\mathcal{C}^*(A, M)).$$

Suppose that A has an augmentation $\varepsilon : A \twoheadrightarrow \mathbb{k}$. Let $\overline{A} := \mathrm{Ker}\varepsilon$ be the augmentation ideal. We denote by $\overline{B}(A) := (Ts\overline{A}, d_0 + d_1)$ the normalized reduced Bar construction, by $\overline{\mathcal{C}}_*(A, M) := (M \otimes T(s\overline{A}), d_0 + d_1)$ the normalized Hochschild chain complex and by $\overline{\mathcal{C}}^*(A, M) :=$

$(\mathrm{Hom}(T(s\overline{A}), M), D_0 + D_1)$ the normalized Hochschild cochain complex.

3. The isomorphism between Hochschild cohomology and Hochschild homology for differential graded algebras

Let A be a differential graded algebra. Let P and Q be two A-bimodules.

The action of $HH^*(A,Q)$ on $HH_*(A,P)$ comes from a (right) action of the $\mathcal{C}^*(A,Q)$ on $\mathcal{C}_*(A,P)$ given by [8, (18) p. 82], [26]

$$\cap : \mathcal{C}_*(A,P) \otimes \mathcal{C}^*(A,Q) \to \mathcal{C}_*(A, P \otimes_A Q)$$

(7)

$$(m[a_1|\dots|a_n], f) \mapsto (m[a_1|\dots|a_n]) \cap f := \sum_{p=0}^{n} \pm (m \otimes_A f[a_1|\dots|a_p])[a_{p+1}|\dots|a_n].$$

Here $\pm$ is the Koszul sign $(-1)^{|f|(|a_1|+\dots|a_n|+n)}$ [33, Proof of Lemma 16].

Let $f : A \to B$ be a morphism of differential graded algebras and let N be a B-bimodule. The linear map $B \otimes_A N \to N$, $b \otimes n \mapsto b.n$ is a morphism of B-bimodules. We call again cap product the composite

$$\mathcal{C}_*(A,B) \otimes \mathcal{C}^*(A,N) \overset{\cap}{\to} \mathcal{C}_*(A, B \otimes_A N) \to \mathcal{C}_*(A,N). \tag{8}$$

In this paper, our goal (statement 9) is to relate the cap product with $B = A$ to the cap product with $N = B = \mathbb{k}$.

Statement 9. *Let A be an augmented differential graded algebra such that each A_i is $\mathbb{k}$-free, $i \in \mathbb{Z}$. Let $c \in HH_d(A,A)$. Denote by $[m] \in Tor_d^A(\mathbb{k},\mathbb{k})$ the image of c by the morphism*

$$HH_d(A,\varepsilon) : HH_d(A,A) \to HH_d(A,\mathbb{k}) = Tor_d^A(\mathbb{k},\mathbb{k}).$$

Suppose that

- *there exists a positive integer n such that for all $i \leq -n$ and for all $i \geq n$, $Tor_i^A(\mathbb{k},\mathbb{k}) = 0$,*
- *each $Tor_i^A(\mathbb{k},\mathbb{k})$ is of finite type, $i \in \mathbb{Z}$,*
- *the morphism of right $Ext_A^*(\mathbb{k},\mathbb{k})$-modules*

$$Ext_A^p(\mathbb{k},\mathbb{k}) \overset{\cong}{\to} Tor_{d-p}^A(\mathbb{k},\mathbb{k}), a \mapsto [m] \cap a$$

is an isomorphism.

Then for any A-bimodule N, the morphism

$$\mathbb{D}^{-1} : HH^p(A,N) \overset{\cong}{\to} HH_{d-p}(A,N), a \mapsto c \cap a$$

is also an isomorphism.

This statement is the Eckmann-Hilton or Koszul dual of [33, Proposition 11]. In this section, we will prove this statement if A is connected. But we wonder if this statement is true in the non-connected case or even for ungraded algebras.

Property 10. Let B and N be two complexes. Consider the natural morphism of complexes $\Theta : B^\vee \otimes N \to \mathrm{Hom}(B,N)$, which sends $\varphi\otimes n$ to the linear map $f : B \to N$ defined by $f(b) := \varphi(b)n$. Suppose that each B_i is $\mathbb{k}$-free.

1) If $B_i = 0$ for all $i \leq -n$ and for all $i \geq n$, for some positive integer n and if each B_i is of finite type or

2) If $H_i(B) = 0$ for all $i \leq -n$ and for all $i \geq n$, for some positive integer n and if each $H_i(B)$ is of finite type

then Θ is a homotopy equivalence.

Proof. 1) Since B is bounded, the component of degre n of $\mathrm{Hom}(B,N)$ is the direct sum $\oplus_{q\in\mathbb{Z}}\mathrm{Hom}(B_{q-n},N_q)$. Since B_{q-n} is free of finite type, $\mathrm{Hom}(B_{q-n},N_q)$ is isomorphic to $B^\vee_{q-n}\otimes N_q$. Therefore Θ is an isomorphism.
2) Since $\mathbb{k}$ is a principal ideal domain, the proof of [36, Lemma 5.5.9] shows that there exists an complex B' satisfying 1) homotopy equivalent to B. By naturality of Θ, Θ is a homotopy equivalence of complexes. □

Lemma 11. *The statement holds whenever N is a trivial A-bimodule, i.e. $a.n = \varepsilon(a)n = n.a$ for $a \in A$ and $n \in N$.*

Proof. Since N is a trivial A-bimodule, the normalized Hochschild chain complex $\overline{\mathcal{C}}_*(A,N)$ is just the tensor product of complexes $\overline{\mathcal{C}}_*(A,\mathbb{k})\otimes N = \overline{B}(A)\otimes N$ (This is also true for the unnormalized Hochschild chain complex, but it is less obvious). And the normalized Hochschild cochain complex $\overline{\mathcal{C}}^*(A,N)$ is just the Hom complex $\mathrm{Hom}(\overline{\mathcal{C}}_*(A,\mathbb{k}),N) = \mathrm{Hom}(\overline{B}(A),N)$. Since the augmentation ideal of A, $\overline{A}$, is $\mathbb{k}$-free, $\overline{B}(A)$ is also $\mathbb{k}$-free. Each $H_i(\overline{B}(A))$ is of finite type and $H_i(\overline{B}(A)) = \mathrm{Tor}_i^A(\mathbb{k},\mathbb{k})$ is null if $i \leq -n$ or $i \geq n$. Therefore by part 2) of Property 10, $\Theta : \overline{B}(A)^\vee\otimes N \overset{\simeq}{\to} \mathrm{Hom}(\overline{B}(A),N)$ is a quasi-isomorphism. A straightforward calculation shows that the following diagram commutes

$$\begin{tikzcd}
\overline{B}(A)^\vee\otimes N \arrow[r, "\Theta", "\simeq"'] \arrow[dr, "([m]\cap -)\otimes N"'] & \mathrm{Hom}(\overline{B}(A),N) = \overline{\mathcal{C}}^*(A,N) \arrow[d, "c\cap -"] \\
& \overline{B}(A)\otimes N = \overline{\mathcal{C}}_*(A,N)
\end{tikzcd}$$

Since $\overline{B}(A)$ is $\Bbbk$-free and its dual $\overline{B}(A)^\vee$ is torsion free, by naturality of Kunneth formula [36, Theorem 5.3.3], $([m]\cap -)\otimes N$ is a quasi-isomorphism. Therefore $c\cap -$ is also a quasi-isomorphism. $\square$

Proposition 12. *Let A be an augmented differential graded algebra. Let N be an A-bimodule. And let $c\in HH_d(A,A)$ satisfying the hypotheses of Statement 9. For any $k\geq 0$, let $F^k := \overline{A^e}^k.N$. Then taking the inverse limit of the cap product with c induces a quasi-isomorphism of complexes*

$$\varprojlim c\cap - : \varprojlim \mathcal{C}^*(A, N/F^k) \xrightarrow{\simeq} \varprojlim \mathcal{C}_*(A, N/F^k).$$

Proof. Consider the augmentation ideal $\overline{A^e}$ of the envelopping algebra A^e. For any $k\geq 0$, let $\overline{A^e}^k$ be the image of the iterated tensor product $\overline{A^e}^{\otimes k}$ by the iterated multiplication of A^e, $\mu : (A^e)^{\otimes k}\to A^e$ and let F^k be the image of $\overline{A^e}^k\otimes N$ by the action $A^e\otimes N\to N$.

The F^k form a decreasing filtration of sub-A-bimodules and sub-complexes of N. Since F^k/F^{k+1} is a trivial A-bimodule, by Lemma 11, the morphism of complexes

$$\mathcal{C}^*(A, F^k/F^{k+1}) \xrightarrow{\simeq} \mathcal{C}_*(A, F^k/F^{k+1}), a\mapsto c\cap a$$

is a quasi-isomorphism. By Noether theorem, we have the short exact sequences of A-bimodules

$$0\to F^k/F^{k+1}\to N/F^{k+1}\to N/F^k\to 0.$$

Since TsA is $\Bbbk$-free, the functors $\mathrm{Hom}_{\Bbbk}(TsA,-)$ and $-\otimes_{\Bbbk} TsA$ preserve short exact sequences. Therefore consider the morphism of short exact sequences of complexes induced by the cap product with c

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{C}^*(A, F^k/F^{k+1}) \arrow[r] \arrow[d, "c\cap -", "\simeq"'] & \mathcal{C}^*(A, N/F^{k+1}) \arrow[r] \arrow[d, "c\cap -"] & \mathcal{C}^*(A, N/F^k) \arrow[r] \arrow[d, "c\cap -"] & 0 \\
0 \arrow[r] & \mathcal{C}_*(A, F^k/F^{k+1}) \arrow[r] & \mathcal{C}_*(A, N/F^{k+1}) \arrow[r] & \mathcal{C}_*(A, N/F^k) \arrow[r] & 0
\end{tikzcd}$$

Using the long exact sequences associated and the five lemma, by induction on k, we obtain that the morphism of complexes

$$\mathcal{C}^*(A, N/F^k) \xrightarrow{\simeq} \mathcal{C}_*(A, N/F^k), a\mapsto c\cap a$$

is a quasi-isomorphism for all $k\geq 0$.

The two towers of complexes

$$\cdots \twoheadrightarrow \mathcal{C}^*(A, N/F^{k+1}) \twoheadrightarrow \mathcal{C}^*(A, N/F^k) \twoheadrightarrow \cdots$$
$$\cdots \twoheadrightarrow \mathcal{C}_*(A, N/F^{k+1}) \twoheadrightarrow \mathcal{C}_*(A, N/F^k) \twoheadrightarrow \cdots$$

satisfy the trivial Mittag-Leffler condition, since all the maps in the two towers are onto. Therefore by naturality of [41, Theorem 3.5.8], for each $p \in \mathbb{Z}$, we have the morphism of short exact sequences induced by the cap product with c

$$\begin{tikzcd}
\varprojlim^1 HH^{p-1}(A, N/F^k) \arrow[r] \arrow[d, "\cong"', "\varprojlim^1 c \cap -"] & H^p \varprojlim \mathcal{C}^*(A, N/F^k) \arrow[r] \arrow[d, "H(\varprojlim c \cap -)"] & \varprojlim HH^p(A, N/F^k) \arrow[d, "\cong"', "\varprojlim c \cap -"] \\
\varprojlim^1 HH_{d+1-p}(A, N/F^k) \arrow[r] & H_{d-p} \varprojlim \mathcal{C}_*(A, N/F^k) \arrow[r] & \varprojlim HH_{d-p}(A, N/F^k)
\end{tikzcd}$$

Using the five Lemma again, we obtain that the middle morphism

$$H(\varprojlim c \cap -) : H^p \varprojlim \mathcal{C}^*(A, N/F^k) \to H_{d-p} \varprojlim \mathcal{C}_*(A, N/F^k)$$

is an isomorphism. □

Corollary 13. *The statement is true if A and N are non-negatively lower graded and $H_0(\varepsilon) : H_0(A) \xrightarrow{\cong} \mathbb{k}$ is an isomorphism.*

Proof. Case 1: We first suppose that $\varepsilon : A_0 \xrightarrow{\cong} \mathbb{k}$ is an isomorphism. Then $\overline{A}^{e^k}$ is concentrated in degres $\geq k$. Therefore F^k and $\mathcal{C}_*(A, F^k)$ are also concentrated in degres $\geq k$. This means that for $n < k$ their components of degre n, $(F^k)_n$ and $[\mathcal{C}_*(A, F^k)]_n$, are trivial. Therefore the tower in degre n

$$\cdots \to (N/F^{k+1})_n \twoheadrightarrow (N/F^k)_n \to \cdots$$

is constant and equal to N_n for $k > n$. This implies that $N_n = \varprojlim (N/F^k)_n$. Therefore as complexes and as A-bimodule, $N = \varprojlim N/F^k$.

Since $\mathcal{C}_*(A, N/F^k)$ is the quotient $\mathcal{C}_*(A, N)/\mathcal{C}_*(A, F^k)$, we also have that as complexes,

$$\mathcal{C}_*(A, N) = \varprojlim \mathcal{C}_*(A, N/F^k)$$

The functor $\mathcal{C}^*(A, -)$ from (differential) A-bimodules to complexes is a right adjoint (to the functor $B(A; A; A) \otimes -$). Therefore $\mathcal{C}^*(A, -)$ preserves inverse limits. Since $N = \varprojlim N/F^k$ in the category of (differential) A-bimodules, we obtain that as complex

$$\mathcal{C}^*(A, N) = \mathcal{C}^*(A, \varprojlim N/F^k) = \varprojlim \mathcal{C}^*(A, N/F^k).$$

Since for any $k \geq 0$, the following square commutes

$$\begin{tikzcd} \mathcal{C}^*(A,N) \arrow[r] \arrow[d, "c\cap -"] & \mathcal{C}^*(A,N/F^k) \arrow[d, "c\cap -"] \\ \mathcal{C}_*(A,N) \arrow[r] & \mathcal{C}_*(A,N/F^k) \end{tikzcd},$$

the quasi-isomorphism given by Proposition 12

$$\lim_{\leftarrow} c\cap - : \lim_{\leftarrow} \mathcal{C}^*(A,N/F^k) \overset{\simeq}{\rightarrow} \lim_{\leftarrow} \mathcal{C}_*(A,N/F^k)$$

coincides with $c\cap - : \mathcal{C}^*(A,N) \to \mathcal{C}_*(A,N)$.

Case 2: Now, we only suppose that $H_0(\varepsilon) : H_0(A) \overset{\cong}{\rightarrow} \mathbb{k}$ is an isomorphism. Let $\tilde{A}$ be the graded $\mathbb{k}$-module defined by $\tilde{A}_0 = \mathbb{k}$, $\tilde{A}_1 = \operatorname{Ker} d : A_1 \to A_0$, $\tilde{A}_n = A_n$ for $n \geq 2$ (Compare with the upper graded version in [11, p. 184]). Clearly $\tilde{A}$ is a $\mathbb{k}$-free sub differential graded algebra of A and the inclusion $j : \tilde{A} \overset{\simeq}{\hookrightarrow} A$ is a quasi-isomorphism since Im $(d : A_1 \to A_0)$ is equal to $\overline{A}_0$.

Since the augmentation ideals of A and $\tilde{A}$, $\overline{A}$ and $\overline{\tilde{A}}$, are $\mathbb{k}$-free and non-negatively lower graded, by [27, 5.3.5] or [10, 4.3(iii)], the three morphisms $HH_*(j,N) : HH_*(\tilde{A},N) \overset{\cong}{\rightarrow} HH_*(A,N)$, $HH^*(j,N) : HH^*(A,N) \overset{\cong}{\rightarrow} HH^*(\tilde{A},N)$ and $HH_*(j,j) : HH_*(\tilde{A},\tilde{A}) \overset{\cong}{\rightarrow} HH_*(A,A)$ are all isomorphims. Let $\tilde{c} \in HH_d(\tilde{A},\tilde{A})$ such that $HH_d(j,j)(\tilde{c}) = c$. Using the definition of the cap product, it is straightforward to check that the following square commutes

$$\begin{tikzcd} HH^*(A,N) \arrow[r, "HH^*(j,N)", "\cong"'] \arrow[d, "c\cap -"] & HH^*(\tilde{A},N) \arrow[d, "\tilde{c}\cap -"] \\ HH_*(A,N) & HH_*(\tilde{A},N) \arrow[l, "\cong"', "HH_*(j,N)"] \end{tikzcd}$$

Let $[\tilde{m}] \in \operatorname{Tor}_d^{\tilde{A}}(\mathbb{k},\mathbb{k})$ such that $\operatorname{Tor}_d^j(\mathbb{k},\mathbb{k})([\tilde{m}]) = [m]$. When $N = \mathbb{k}$, the previous square specializes to the following commutative square

$$\begin{tikzcd} \operatorname{Ext}_A^*(\mathbb{k},\mathbb{k}) \arrow[r, "\operatorname{Ext}_j^*(\mathbb{k},\mathbb{k})", "\cong"'] \arrow[d, "{[m]\cap -}"', "\cong"] & \operatorname{Ext}_{\tilde{A}}^*(\mathbb{k},\mathbb{k}) \arrow[d, "{[\tilde{m}]\cap -}"] \\ \operatorname{Tor}_*^A(\mathbb{k},\mathbb{k}) & \operatorname{Tor}_*^{\tilde{A}}(\mathbb{k},\mathbb{k}) \arrow[l, "\cong"', "\operatorname{Tor}_*^j(\mathbb{k},\mathbb{k})"] \end{tikzcd}$$

By hypothesis, $[m]\cap -$ is an isomorphism. Therefore $[\tilde{m}]\cap -$ is also an isomorphism. So since $\tilde{A}_0 = \mathbb{k}$, we have seen in the case 1, that

$$\tilde{c}\cap - : HH^*(\tilde{A},N)\stackrel{\cong}{\to} HH_*(\tilde{A},N)$$

is an isomorphism. Therefore

$$c\cap - : HH^*(A,N)\stackrel{\cong}{\to} HH_*(A,N)$$

is also an isomorphism. □

Corollary 14. *The statement is true if A and N are non-negatively upper graded, $H^0(\varepsilon) : H^0(A)\stackrel{\cong}{\to}\mathbb{k}$ is an isomorphism and $\mathbb{k}$ is a field.*

Proof. Case 1: We first suppose that $\varepsilon : A^0\stackrel{\cong}{\to}\mathbb{k}$ is an isomorphism. Since $T(sA)$ has non-trivial elements of negative degres, we need to use the normalized Hochschild chain and cochain complexes $\overline{\mathcal{C}}_*$ and $\overline{\mathcal{C}}^*$ instead of the unnormalized $\mathcal{C}_*$ and $\mathcal{C}^*$. Now the proof is the same as in Case 1 of the proof of Corollary 13.

Case 2: Now, we only suppose that that $H^0(\varepsilon) : H^0(A)\stackrel{\cong}{\to}\mathbb{k}$ is an isomorphism. Since $\mathbb{k}$ is a field, by [11, p. 184]), there exists a differential graded algebra $\tilde{A}$, non-negatively upper graded, equipped with a quasi-isomorphism $j : \tilde{A}\stackrel{\simeq}{\to} A$ such that $\tilde{A}^0=\mathbb{k}$. Now the rest of the proof is exactly the same as in Case 2 of the proof of Corollary 13. □

4. Comparison of the Cap products in Hochschild and group (co)homology

Let G be a discrete group. Let M and N be two $\mathbb{k}[G]$-bimodules. Let $\eta : \mathbb{k}\to\mathbb{k}[G]$ be the unit map. Let $E : \mathbb{k}[G]\to\mathbb{k}[G\times G^{op}]$ be the morphism of algebras mapping g to (g,g^{-1}). Let

$$\tilde{\eta} : \mathbb{k}[G\times G^{op}]\otimes_{\mathbb{k}[G]}\mathbb{k}\to\mathbb{k}[G]$$

be the unique morphism of left $\mathbb{k}[G\times G^{op}]$-modules extending η. Since $\mathbb{k}[G\times G^{op}]$ is flat as left $\mathbb{k}[G]$-module via E and since $\tilde{\eta}$ is an isomorphism, by Eckmann-Schapiro [22, Chapt IV.Proposition 12.2], we obtain the well-known isomorphisms between Hochschild (co)homology and group (co)homology:

$$\mathrm{Ext}^*_E(\eta,N) : HH^*(\mathbb{k}[G],N)=\mathrm{Ext}^*_{\mathbb{k}[G\times G^{op}]}(\mathbb{k}[G],N)\stackrel{\cong}{\to}\mathrm{Ext}^*_{\mathbb{k}[G]}(\mathbb{k},\tilde{N})=H^*(G,\tilde{N}).$$

and

$$\mathrm{Tor}^E_*(M,\eta) : H_*(G,\tilde{M})=\mathrm{Tor}^{\mathbb{k}[G]}_*(\tilde{M},\mathbb{k})\stackrel{\cong}{\to}\mathrm{Tor}^*_{\mathbb{k}[G\times G^{op}]}(M,\mathbb{k}[G])=HH_*(\mathbb{k}[G],M).$$

Here $\tilde{M}$ and $\tilde{N}$ denote the $\mathbb{k}[G]$-modules obtained by restriction of scalar via E. Note that we regard any left $\mathbb{k}[G]$-module as an right $\mathbb{k}[G]$-module via $g\mapsto g^{-1}$ [4, p. 55].

Proposition 15. *Remark that the canonical surjection*

$$q : \tilde{M} \otimes \tilde{N} \twoheadrightarrow M \widetilde{\otimes_{\Bbbk[G]}} N$$

is a morphism of $\Bbbk[G]$*-modules, since* $q(gmg^{-1} \otimes gng^{-1}) = gm \otimes ng^{-1}$.

i) Cup product $\cup$ *in Hochschild cohomology versus cup product in group cohomology (slight extension of* [35, Proposition 3.1]*). The following diagram commutes*

$$\begin{tikzcd}
HH^*(\Bbbk[G], M) \otimes HH^*(\Bbbk[G], N) \arrow[rr, "\cup"] \arrow[d, "Ext^*_E(\eta,M)\otimes Ext^*_E(\eta,N)"'] & & HH^*(\Bbbk[G], M \otimes_{\Bbbk[G]} N) \arrow[d, "Ext^*_E(\eta, M\otimes_{\Bbbk[G]} N)"'] \\
H^*(G, \tilde{M}) \otimes H^*(G, \tilde{N}) \arrow[r, "\cup"'] & H^*(G, \tilde{M} \otimes \tilde{N}) \arrow[r, "H^*(G,q)"'] & H^*(G, M \widetilde{\otimes_{\Bbbk[G]}} N)
\end{tikzcd}$$

ii) Cap products $\cap$*. The following diagram commutes*

$$\begin{tikzcd}
HH_*(\Bbbk[G], M) \otimes HH^*(\Bbbk[G], N) \arrow[rr, "\cap"] & & HH_*(\Bbbk[G], M \otimes_{\Bbbk[G]} N) \\
H_*(G, \tilde{M}) \otimes H^*(G, \tilde{N}) \arrow[r, "\cap"'] \arrow[u, "Tor^E_*(M,\eta)\otimes Ext^*_E(\eta,N)^{-1}"] & H_*(G, \tilde{M} \otimes \tilde{N}) \arrow[r, "H_*(G,q)"'] & H_*(G, M \widetilde{\otimes_{\Bbbk[G]}} N) \arrow[u, "Tor^E_*(M\otimes_{\Bbbk[G]} N,\eta)"']
\end{tikzcd}$$

Remark 16. In the case $N = \Bbbk[G]$ [35, (3.3)], the morphism of $\Bbbk[G]$-modules $q : \tilde{M} \otimes \Bbbk[\tilde{G}] \twoheadrightarrow M \widetilde{\otimes_{\Bbbk[G]} \Bbbk[G]} \cong \tilde{M}$ is simply the action $m \otimes g \mapsto m.g$.

In the case $M = N = \Bbbk[G]$, the diagram i) in Proposition 15 means that

$$\mathrm{Ext}^*_E(\eta, \Bbbk[G]) : HH^*(\Bbbk[G], \Bbbk[G]) \to H^*(G, \Bbbk[\tilde{G}])$$

is a morphism of graded algebras.

In the case $N = \Bbbk[G]$, the diagram ii) means that

$$\mathrm{Tor}^E_*(M, \eta) : H_*(G, \tilde{M}) \to HH_*(\Bbbk[G], M)$$

is a morphism of right $HH^*(\Bbbk[G], \Bbbk[G])$-modules:

$$\mathrm{Tor}^E_*(\eta, \Bbbk[G]) \left(\alpha \cap \mathrm{Ext}^*_E(\eta, \Bbbk[G])(\varphi)\right) = \mathrm{Tor}^E_*(\eta, \Bbbk[G])(\alpha) \cap \varphi$$

for any $\alpha \in H_*(G, \tilde{M})$ and any $\varphi \in HH^*(\Bbbk[G], \Bbbk[G])$.

Proof. Siegel and Witherspoon [35, Proposition 3.1] proved i) using that for any projective resolution P of $\Bbbk$ as left $\Bbbk[G]$-modules,

$$X := \Bbbk[G \times G^{op}] \otimes_{\Bbbk[G]} P$$

is a projective resolution of $\Bbbk[G]$ as $\Bbbk[G]$-bimodules. Let $\iota : P \hookrightarrow \tilde{X}$ the left $\Bbbk[G]$-linear map defined by $\iota(x) = (1, 1) \otimes x$. Using that

$$\mathrm{Hom}_E(\iota, N) : \mathrm{Hom}_{\Bbbk[G\times G^{op}]}(X, N) \xrightarrow{\cong} \mathrm{Hom}_{\Bbbk[G]}(P, \tilde{N})$$

is an isomorphism of complexes inducing $\mathrm{Ext}^*_E(\eta, N)$ and that

$$M \otimes_E \iota : \tilde{M} \otimes_{\Bbbk[G]} P \overset{\cong}{\to} M \otimes_{\Bbbk[G\times G^{op}]} X$$

is an isomorphism of complexes inducing $\mathrm{Tor}^E_*(M, \eta)$, Siegel and Witherspoon [35, Proposition 3.1] proved i). But one can also prove similarly ii).

We find more simple to give a proof of ii) using the Bar resolution. Let $\iota : B(\Bbbk[G];\Bbbk[G];\Bbbk) \to B(\Bbbk[G];\widetilde{\Bbbk[G]};\Bbbk[G])$ be the linear map defined by

$$\iota(g_0[g_1|\cdots|g_n]) = g_0[g_1|\cdots|g_n]g_n^{-1}\dots g_0^{-1}.$$

Obviously ι fits into the commutative diagram of left $\Bbbk[G]$-modules

$$\begin{tikzcd}
B(\Bbbk[G];\widetilde{\Bbbk[G]};\Bbbk[G]) \arrow[r] & \Bbbk[G] \\
B(\Bbbk[G];\Bbbk[G];\Bbbk) \arrow[r] \arrow[u, dotted, "\iota"] & \Bbbk \arrow[u, "\eta"']
\end{tikzcd}$$

A straightforward computation shows that ι is a morphism of complexes. Therefore $\mathrm{Hom}_E(\iota, N)$ is an morphism of complexes from $\mathcal{C}^*(\Bbbk[G], N) \cong (\mathrm{Hom}_{\Bbbk[G\times G^{op}]}(B(\Bbbk[G];\Bbbk[G];\Bbbk[G]), N)$ to $\mathrm{Hom}_{\Bbbk[G]}(B(\Bbbk[G];\Bbbk[G];\Bbbk), \tilde{N})$. inducing $\mathrm{Ext}^*_E(\eta, N)$ and $M \otimes_E \iota$ is an morphism of complexes from

$$B(\tilde{M};\Bbbk[G];\Bbbk) \cong \tilde{M} \otimes_{\Bbbk[G]} B(\Bbbk[G];\Bbbk[G];\Bbbk)$$

to

$$M \otimes_{\Bbbk[G\times G^{op}]} B(\Bbbk[G];\Bbbk[G];\Bbbk[G]) \cong \mathcal{C}_*(\Bbbk[G], M)$$

inducing $\mathrm{Tor}^E_*(M, \eta)$. Explicitly $M \otimes_E \iota$ is the morphism of complexes

$$\xi : B(\tilde{M};\Bbbk[G];\Bbbk) \to \mathcal{C}_*(\Bbbk[G], M)$$

defined by [14, (2.20)]

$$\xi(m[g_1|\dots|g_n] = g_n^{-1}\dots g_1^{-1}m[g_1|\dots|g_n]. \tag{17}$$

And $\mathrm{Hom}_E(\iota, N) : \mathcal{C}^n(\Bbbk[G], N) \to \mathrm{Hom}(\Bbbk[G]^{\otimes n}, \tilde{N})$ is the linear map ξ mapping $\varphi \in \mathcal{C}^n(\Bbbk[G], N)$ to the linear map $\xi(\varphi) : \Bbbk[G]^{\otimes n} \to \tilde{N}$ defined by

$$\xi(\varphi)([g_1|\dots|g_n]) = \varphi([g_1|\dots|g_n])g_n^{-1}\dots g_1^{-1}.$$

Both $M \otimes_E \iota$ and $\mathrm{Hom}_E(\iota, N)$ are in fact isomorphisms of complexes. The inverse of $M \otimes_E \iota$ is the morphism of complexes $\Phi : \mathcal{C}_*(\Bbbk[G], M) \to B(\tilde{M};\Bbbk[G];\Bbbk)$ defined by [27, 7.4.2.1]

$$\Phi(m[g_1|\dots|g_n]) = g_1\dots g_n m[g_1|\dots|g_n].$$

Let F be any projective resolution of $\Bbbk$ as left $\Bbbk[G]$-module. Let P and Q be two $\Bbbk[G]$-modules. The cap product in group cohomology is the composite [4, p. 113], denoted $\cap$

$$\begin{tikzcd}
P \otimes_{\Bbbk[G]} F \otimes \mathrm{Hom}_{\Bbbk[G]}(F, Q) \arrow[d, "Id\otimes_{\Bbbk[G]}\Delta\otimes_{\Bbbk[G]}Id"] \\
P \otimes_{\Bbbk[G]} (F \otimes F) \otimes \mathrm{Hom}_{\Bbbk[G]}(F, Q) \arrow[d, "\gamma"] \\
(P \otimes Q) \otimes_{\Bbbk[G]} F
\end{tikzcd}$$

where $\gamma(a\otimes x\otimes y\otimes u) = (-1)^{|u||x|+|u||y|} a\otimes u(x)\otimes y$ and Δ is a *diagonal approximation*. In the case, F is the Bar resolution $B(\Bbbk[G];\Bbbk[G];\Bbbk)$, one can take Δ to be the Alexander-Whitney map

$$AW : B(\Bbbk[G];\Bbbk[G];\Bbbk) \to B(\Bbbk[G];\Bbbk[G];\Bbbk) \otimes B(\Bbbk[G];\Bbbk[G];\Bbbk)$$

defined by [4, p. 108 (1.4)]:

$$AW(g_0[g_1|\dots|g_n]) = \sum_{p=0}^{n} g_0[g_1|\dots|g_p] \otimes g_0\dots g_p[g_{p+1}|\dots|g_n].$$

Therefore the cap product

$$\cap : B(P;\Bbbk[G];\Bbbk) \otimes \mathrm{Hom}(B(\Bbbk[G]), Q), d \to B(P\otimes Q;\Bbbk[G];\Bbbk)$$

is the morphism of complexes mapping $m[g_1|\dots|g_n] \otimes u : G^p \to Q$ to $m.g_1\dots g_p \otimes u(g_1,\dots,g_p).g_1\dots g_p[g_{p+1}|\dots|g_n]$. Using the explicit formula (7) for the cap product in Hochschild cohomology, it is easy to check that the following diagram commutes

$$\begin{tikzcd}
\mathcal{C}_*(\Bbbk[G], M) \otimes \mathcal{C}^*(\Bbbk[G], N) \arrow[rr, "\cap"] \arrow[d, "\Phi\otimes\mathrm{Hom}_E(\iota,N)"] & & \mathcal{C}_*(\Bbbk[G], M \otimes_{\Bbbk[G]} N) \arrow[d, "\Phi"] \\
B(\tilde{M};\Bbbk[G];\Bbbk) \otimes B(\tilde{N};\Bbbk[G];\Bbbk) \arrow[r, "\cap"'] & B(\tilde{M}\otimes\tilde{N};\Bbbk[G];\Bbbk) \arrow[r, "B(q;\Bbbk[G];\Bbbk)"'] & B(\widetilde{M \otimes_{\Bbbk[G]} N};\Bbbk[G];\Bbbk)
\end{tikzcd}$$

By applying homology, ii) is proved. $\square$

Definition 18. [27, 7.4.5 when $z = 1$] Let $\sigma : B(\Bbbk[G]) \hookrightarrow \mathcal{C}_*(\Bbbk[G], \Bbbk[G])$ be the linear map defined by

$$\sigma([g_1|\dots|g_n]) = g_n^{-1}\dots g_1^{-1}[g_1|\dots|g_n].$$

Property 19. i) [27, 7.4.5 when z=1] The map σ is a morphism of cyclic modules.

ii) The morphism of complexes σ coincides with the composite

$$B(\Bbbk[G]) \overset{B(\eta;\Bbbk[G];\Bbbk)}{\to} B(\widetilde{\Bbbk[G]};\Bbbk[G];\Bbbk) \underset{\cong}{\overset{\xi}{\to}} \mathcal{C}_*(\Bbbk[G];\Bbbk[G]).$$

Here ξ is the isomorphism of complexes defined by (17). Note that the unit map $\eta : \Bbbk \to \widetilde{\Bbbk[G]}$ is a morphism of $\Bbbk[G]$-modules.

iii) In particular, in homology, σ coincides with

$$\mathrm{Tor}^E(\eta,\eta) : H_*(G;\Bbbk) \to HH_*(\Bbbk[G];\Bbbk[G]).$$

iv) The map σ is a section of

$$\mathcal{C}_*(\Bbbk[G],\varepsilon) : \mathcal{C}_*(\Bbbk[G],\Bbbk[G]) \to \mathcal{C}_*(\Bbbk[G],\Bbbk) = B(\Bbbk[G]).$$

Corollary 20. *Let G be any discrete group. Let N be a $\Bbbk[G]$-bimodule. Let $\sigma : H_*(G;\Bbbk) \to HH_*(\Bbbk[G];\Bbbk[G])$ be the section of $HH_*(\Bbbk[G],\varepsilon) : HH_*(\Bbbk[G],\Bbbk[G]) \to H_*(G,\Bbbk)$ defined in Definition 18. Let $z \in H_d(G,\Bbbk)$ be any element in group homology. Then the following square commutes*

$$\begin{tikzcd}
H^p(G,\tilde{N}) \arrow[r, "z\cap -"] & H_{d-p}(G,\tilde{N}) \arrow[d, "\cong"', "Tor^E_*(N,\eta)"] \\
HH^p(\Bbbk[G],N) \arrow[u, "Ext^*_E(\eta,N)", "\cong"'] \arrow[r, "\sigma(z)\cap -"] & HH_{d-p}(\Bbbk[G],N)
\end{tikzcd}$$

Proof.

$$\begin{tikzcd}
HH_*(\Bbbk[G],\Bbbk[G]) \otimes HH^*(\Bbbk[G],N) \arrow[rr, "\cap"] & & HH_*(\Bbbk[G],\Bbbk[G]\otimes_{\Bbbk[G]} N) \\
H_*(G,\widetilde{\Bbbk[G]}) \otimes H^*(G,\tilde{N}) \arrow[u, "\mathrm{Tor}^E_*(\Bbbk[G],\eta)\otimes\mathrm{Ext}^*_E(\eta,N)^{-1}"] \arrow[r, "\cap"'] & H_*(G,\widetilde{\Bbbk[G]}\otimes\tilde{N}) \arrow[r, "H_*(G,q)"'] & H_*(G,\tilde{N}) \arrow[u, "\mathrm{Tor}^E_*(N,\eta)"'] \\
H_*(G,\Bbbk) \otimes H^*(G,\tilde{N}) \arrow[u, "H_*(G,\eta)\otimes Id"] \arrow[r, "\cap"'] & H_*(G,\Bbbk\otimes\tilde{N}) \arrow[u, "H_*(G,\eta\otimes\tilde{N})"] \arrow[ur, "\cong"'] &
\end{tikzcd}$$

The top rectangle commutes by ii) of Proposition 15 in the case $M = \Bbbk[G]$. The bottom square commutes by naturality of the cap product in group (co)homology with respect to the morphism of $\Bbbk[G]$-modules $\eta : \Bbbk \to \widetilde{\Bbbk[G]}$. The bottom triangle commutes by functoriality of $H_*(G,-)$. By ii) or iii) of Property 19, the vertical composite is

$$\sigma\otimes\mathrm{Ext}^*_E(\eta,N)^{-1} : H_*(G,\Bbbk)\otimes H^*(G,\tilde{N}) \to HH_*(\Bbbk[G],\Bbbk[G])\otimes HH^*(\Bbbk[G],N).$$

$\square$

5. A new definition of Batalin-Vilkovisky algebras

Definition 21. A *Gerstenhaber algebra* is a commutative graded algebra A equipped with a linear map $\{-,-\}: A_i\otimes A_j\to A_{i+j+1}$ of degree 1 such that:
a) the bracket $\{-,-\}$ gives A a structure of graded Lie algebra of degree 1. This means that for each a, b and $c\in A$

$$\{a,b\}=-(-1)^{(|a|+1)(|b|+1)}\{b,a\} \text{ and} \tag{22}$$

$$\{a,\{b,c\}\}=\{\{a,b\},c\}+(-1)^{(|a|+1)(|b|+1)}\{b,\{a,c\}\}.$$

b) the product and the Lie bracket satisfy the following relation called the Poisson relation:

$$\{a,bc\}=\{a,b\}c+(-1)^{(|a|+1)|b|}b\{a,c\}.$$

Definition 23. A *Batalin-Vilkovisky algebra* is a Gerstenhaber algebra A equipped with a degree 1 linear map $\Delta: A_i\to A_{i+1}$ such that $\Delta\circ\Delta=0$ and such that the bracket is given by

$$\{a,b\}=(-1)^{|a|}\left(\Delta(a\cup b)-(\Delta a)\cup b-(-1)^{|a|}a\cup(\Delta b)\right) \tag{24}$$

for a and $b\in A$.

Remark 25. In (24), a sign (here the sign chosen is $(-1)^{|a|}$) is needed (See [25, (1.6)] or [17, beginning of the proof of Proposition 1.2]), since the Lie bracket of degre $+1$ is graded antisymmetric (equation (22))while the associative product is graded commutative. Therefore the definition of Batalin-Vilkovisky algebra in [18, Theorem 3.4.3 (ii)] and [26, p. 1] has a sign problem.

The following characterization of Batalin-Vilkovisky algebras was proved by Koszul and rediscovered by Getzler and by Penkava and Schwarz.

Proposition 26. [25, p. 3] [17, Proposition 1.2] [34] *Let A be a commutative graded algebra A equipped with an operator $\Delta: A_i\to A_{i+1}$ of degree 1 such that $\Delta\circ\Delta=0$. Consider the bracket $\{\ ,\ \}$ of degree $+1$ defined by*

$$\{a,b\}=(-1)^{|a|}\left(\Delta(ab)-(\Delta a)b-(-1)^{|a|}a(\Delta b)\right)$$

for any a, $b\in A$. Then A is a Batalin-Vilkovisky algebra if and only if Δ is a differential operator of degree ≤ 2, this means that for a, b and $c\in A$,

$$\begin{aligned}\Delta(abc)=\Delta(ab)c+(-1)^{|a|}a\Delta(bc)+(-1)^{(|a|-1)|b|}b\Delta(ac)\\ -(\Delta a)bc-(-1)^{|a|}a(\Delta b)c-(-1)^{|a|+|b|}ab(\Delta c).\end{aligned} \tag{27}$$

Remark that till now, in this section, it is not necessary that the algebras have an unit. Now if the algebras have an unit, we give a new characterization of Batalin-Vilkovisky algebra. One implication in this new characterization is inspired by Ginzburg's proof of Proposition 32. As we will recall in the proof of Theorem 66, the converse in this characterization is due to [24, "the restriction of this derived bracket to A is the BV-bracket", p. 1270].

Proposition 28. *Let A be a Gerstenhaber algebra A equipped with an operator $\Delta : A \to A$ of degree 1 such that $\Delta \circ \Delta = 0$. For any $a \in A$, denote by $l_a : A \to A$, the left multiplication by a, explicitly $l_a(b) = ab$, $b \in A$. Denote by $[f, g] = f \circ g - (-1)^{|f||g|} g \circ g$ the graded commutator of two endomorphisms $f : A \to A$ and $g : A \to A$. Then A is a Batalin-Vilkovisky algebra if and only if for a, $b \in A$,*

$$l_{\{a,b\}} = -[[l_a, \Delta], l_b] \quad \text{and} \quad \Delta(1) = 0.$$

Proof. For a and $b \in A$,

$$\begin{aligned}[][[l_a, \Delta], l_b] = &\left(l_a \circ \Delta - (-1)^{|a}\Delta \circ l_a\right) \circ l_b \\ &- (-1)^{|b|(|a|+1)} l_b \circ \left(l_a \circ \Delta - (-1)^{|a}\Delta \circ l_a\right) \\ = l_a \circ \Delta \circ l_b - (-1)^{|a|}\Delta \circ l_{ab} &- (-1)^{|b|} l_{ab} \circ \Delta + (-1)^{|b|(|a|+1)+|a|} l_b \circ \Delta \circ l_a.\end{aligned}$$

Therefore by applying this equality of operators to $c \in A$, we have

$$\begin{aligned}(29) \quad -(-1)^{|a|}[[l_a, \Delta], l_b](c) = &-(-1)^{|a|} a\Delta(bc) + \Delta(abc) \\ &+ (-1)^{|a|+|b|} ab\Delta(c) - (-1)^{|b|(|a|+1)} b\Delta(ac).\end{aligned}$$

Suppose that A is a Batalin-Vilkovisky algebra. By Proposition 26, using (29), we obtain that

$$-(-1)^{|a|}[[l_a, \Delta], l_b](c) = \Delta(ab)c - (\Delta a)bc - (-1)^{|a|} a(\Delta b)c = (-1)^{|a|}\{a, b\}c.$$

Therefore $-[[l_a, \Delta], l_b] = l_{\{a,b\}}$. In the case $a = b = c = 1$, equation (27) gives $\Delta(1) = 3\Delta(1) - 3\Delta(1) = 0$.

Conversely, suppose that $\Delta(1) = 0$ and that $l_{\{a,b\}} = -[[l_a, \Delta], l_b]$. Then using (29)

$$\{a, b\} = l_{\{a,b\}}(1) = (-1)^{|a|}\left(-(-1)^{|a|} a\Delta(b) + \Delta(ab) + 0 - (\Delta a)b\right).$$

Therefore, by Definition 23, A is a Batalin-Vilkovisky algebra. $\square$

6. Batalin-Vilkovisky algebra structures on Hochschild cohomology

Let A be a differential graded algebra. The cap product defined in Section 3,

$$HH_*(A,A)\otimes HH^*(A,A)\overset{\cap}{\to} HH_*(A,A), c\otimes a\mapsto c\cap a$$

is a right action.

Following Tsygan definition of a calculus, we want a left action. Therefore, we define as in [26, Definition 1.2],

$$\mathcal{C}^*(A,A)\otimes\mathcal{C}_*(A,A)\to\mathcal{C}_*(A,A)$$

$$f\otimes c\mapsto i_f(c)=f\cdot c:=(-1)^{|c||f|}c\cap f. \tag{30}$$

Explicitly

$$i_f(m[a_1|\dots|a_n]):=\sum_{p=0}^{n}(-1)^{|m||f|}(m.f[a_1|\dots|a_p])[a_{p+1}|\dots|a_n].$$

The sign in [8, (18) p. 82] is different. But with our choice of signs, we recover Proposition 2.6 in [8, p. 82]. Indeed for D, $E\in\mathcal{C}^*(A,A)$ and $c\in\mathcal{C}_*(A,A)$,

$$\begin{aligned}D\cdot(E\cdot c)&=(-1)^{|c||E|}D\cdot(c\cap E)=(-1)^{|c||E|+|D||c|+|D||E|}(c\cap E)\cap D\\&=(-1)^{|c||E|+|D||c|+|D||E|}c\cap(E\cup D)=(-1)^{|D||E|}(E\cup D)\cdot c\end{aligned}$$

Since the cup product on $HH^*(A,A)$ is graded commutative, for D, $E\in HH^*(A,A)$ and $c\in HH_*(A,A)$, we have

$$D\cdot(E\cdot c)=(D\cup E)\cdot c, \tag{31}$$

i. e. a left action. Note that in [33], the author forgot to twist the right action by the sign $(-1)^{|c||f|}$, therefore has also a sign problem!

Proposition 32. [18, Theorem 3.4.3 (ii)] *Let $c\in HH_d(A,A)$ such that the morphism of left $HH^*(A,A)$-modules*

$$HH^p(A,A)\overset{\cong}{\to} HH_{d-p}(A,A),\ a\mapsto a\cdot c$$

is an isomorphism. If $B(c)=0$ then the Gerstenhaber algebra $HH^*(A,A)$ equipped with $-B$ is a Batalin-Vilkovisky algebra.

Proof. Let us rewrite the proof of Victor Ginzburg (or more precisely the proof, we already gave in [33, Proposition 13 and Lemma 15]) using explicitly our Proposition 28 and our choice of signs. Denote by

$$HH^p(A,A)\otimes HH_j(A,A)\to HH_{j-p+1}(A,A)$$

$$a\otimes x\mapsto L_a(x)$$

the action of the suspended graded Lie algebra $sHH^*(A,A)$ on $HH_*(A,A)$. Gelfand, Daletski and Tsygan [15] proved that the Gerstenhaber algebra $HH^*(A,A)$ and Connes boundary map B on $HH_*(A,A)$ form a calculus [8, p. 93]. In particular, we have the two relations

$$L_a = [B, i_a]$$

and [8, Proposition 2.9 p. 83]

$$i_{\{a,b\}} = (-1)^{|a|+1}[L_a, i_b]. \tag{33}$$

Therefore

$$i_{\{a,b\}} = (-1)^{|a|+1}[[B, i_a], i_b] = [[i_a, B], i_b]. \tag{34}$$

The operator Δ on $HH^*(A,A)$ is defined by

$$(\Delta a)\cdot c := -B(a\cdot c) \quad \text{for any} \quad a \in HH^*(A,A).$$

Thus $B(c) = 0$ implies $\Delta(1) = 0$. Since we have a left action (equation (31)), $l_a(b)\cdot c = (a\cup b)\cdot c = a\cdot(b\cdot c) = i_a(b\cdot c)$ and so equation (34) is equivalent to

$$l_{\{a,b\}} = -[[l_a, \Delta], l_b].$$

Therefore, by Proposition 28, $HH^*(A,A)$ is a Batalin-Vilkovisky algebra. □

Remark 35. (Signs)

i) In [8, Example 4.6 p. 93], Tsygan writes that it follows from [8, 2.9 p. 83], that $i_{\{a,b\}} = [L_a, i_b]$. As Tsygan [39] has kindly confirmed us, there should be a sign in this formula: from [8, 2.9 p. 83], the correct equation with the signs is equation (33) above or equivalently $i_{\{a,b\}} = [i_a, L_b]$ [38, (0.1)].

ii) In a calculus, there is a third relation, that we do not use in this paper:

$$L_{ab} = L_a i_b + (-1)^{|a|} i_a L_b.$$

Since $ab = (-1)^{|a||b|}ba$,

$$L_{ab} = (-1)^{|a||b|}L_{ba} = (-1)^{|a||b|}L_b i_a + (-1)^{(|a|+1)|b|} i_b L_a$$

and therefore

$$[L_a, i_b] = (-1)^{|a||b|}[L_b, i_a]. \tag{36}$$

Since $\{a,b\} = -(-1)^{(|a|+1)(|b|+1)}\{b,a\}$,

-if we suppose like in [8, Example 4.6 p. 93] that $i_{\{a,b\}} = [L_a, i_b]$, we obtain that

$$[L_a, i_b] = -(-1)^{(|a|+1)(|b|+1)}[L_b, i_a]. \tag{37}$$

The two equations (36) and (37) seem incoherent. Therefore the definition of calculus in [8, Definition 4.3 p. 93] has some sign problem.

-on the contrary, if we suppose (33), we obtain again (36).

7. Proof of the main theorem for path-connected groups

Cap products associated to coalgebras. Let C be a (differential graded) coalgebra. Then its dual $C^\vee$ is a (differential graded) algebra. Let N be a left C-comodule. Denote by $\Delta_N : N \to C \otimes N$ the structure map. Let $\cap : N \otimes C^\vee \to N$ be the composite

$$(38)\quad N \otimes C^\vee \overset{\Delta_N \otimes C^\vee}{\to} C \otimes N \otimes C^\vee \overset{C \otimes \tau}{\to} C \otimes C^\vee \otimes N \overset{ev \otimes N}{\to} \Bbbk \otimes N \cong N.$$

Here τ denotes the twist map given by $n \otimes \varphi \mapsto (-1)^{|n||\varphi|}\varphi \otimes n$ and ev is the evaluation map defined by $ev(c \otimes \varphi) = (-1)^{|\varphi||c|}\varphi(c)$. Then N equipped with the cap product is a right $C^\vee$-module [37, Proposition 2.1.1]. In this paper, we are only interested in the case $N = C$.

Example 39. Let X be any topological space. The (normalized or unnormalized) singular chains of X, $S_*(X)$ forms a differential graded coalgebra [30, p. 244-5]. The cap product defined by (38) associated to $C = S_*(X)$, $\cap : S_*(X) \otimes S^*(X) \to S_*(X)$ is the usual cap product.

Example 40. Let A be any augmented differential graded algebra. Then the reduced (normalized or not) Bar construction $B(A) = \mathcal{C}_*(A, \Bbbk)$ is a differential graded coalgebra. The diagonal $\Delta : B(A) \to B(A) \otimes B(A)$ is given by

$$\Delta([a_1| \dots |a_n]) = \sum_{p=0}^{n} [a_1| \dots |a_p] \otimes [a_{p+1}| \dots |a_n].$$

The cap product defined by (38) associated to $C = B(A)$ is given by

$$\cap : B(A) \otimes B(A)^\vee \to B(A)$$

$$[a_1| \dots |a_n] \cap f = \sum_{p=0}^{n} (-1)^{|f|(|a_1|+\cdots+|a_n|+n)} f([a_1| \dots |a_p])[a_{p+1}| \dots |a_n].$$

Therefore this cap product coincides with the cap product on the Hochschild (co)chain complex $\cap : \mathcal{C}_*(A, \Bbbk) \otimes \mathcal{C}^*(A, \Bbbk) \to \mathcal{C}_*(A, \Bbbk)$ defined by (8) in the case $N = B = \Bbbk$.

Proposition 41. *Let $f : C \overset{\simeq}{\to} D$ be a quasi-isomorphism of coalgebras. Suppose that C and D are $\Bbbk$-free. Let $\tilde{c} \in C$ and $\tilde{d} \in D$ such that $\tilde{d} = H_*(f)([\tilde{c}])$. Consider the cap products defined by (38) associated to the coalgebras C and D. Then*

the morphism of right $C^\vee$-modules $\tilde{c}\cap - : C^\vee \to C$ given by $a \mapsto \tilde{c}\cap a$ is a quasi-isomorphism if and only if *the morphism of right $D^\vee$-modules $\tilde{d}\cap - : D^\vee \to D$ given by $a \mapsto \tilde{d}\cap a$ is a quasi-isomorphism.*

Proof. The transpose of f: $f^\vee : D^\vee \to C^\vee$ is a morphism of differential graded algebras. Therefore $f^\vee$ is a morphism of right $D^\vee$-modules. Dually, since f is a morphism of coalgebras, f is a morphism of left D-comodules and therefore f is also a morphism of right $D^\vee$-modules by (38), i. e. $f(c \cap f^\vee(\varphi)) = f(c) \cap \varphi$ for any $c \in C$ and $\varphi \in D^\vee$. Note that if f is the coalgebra map $S_*(\lambda) : S_*(X) \to S_*(Y)$ induced by a continuous map $\lambda : X \to Y$, this formula is well known ([3, Chapter VI 5. Theorem (4)] or [21, p. 241]).

The composite of the morphisms of right $D^\vee$-modules

$$D^\vee \overset{f^\vee}{\to} C^\vee \overset{\tilde{c}\cap -}{\to} C \overset{f}{\to} D$$

maps 1 to $f(\tilde{c})$ and therefore coincides with the morphism of right $D^\vee$-modules $D^\vee \to D$, $a \mapsto f(\tilde{c}) \cap a$. Since $[\tilde{d}] = [f(\tilde{c})]$, the two maps $a \mapsto f(\tilde{c}) \cap a$ and $a \mapsto \tilde{d} \cap a$ coincide after passing to homology. Therefore after passing to homology, the following square commutes

$$\begin{tikzcd} D^\vee \arrow[r, "f^\vee"] \arrow[d, "\tilde{d}\cap -"'] & C^\vee \arrow[d, "\tilde{c}\cap -"] \\ D & C \arrow[l, "f", "\simeq"'] \end{tikzcd} \tag{42}$$

Since both C and D are $\Bbbk$-free and $\Bbbk$ is a principal ideal domain, by naturality of the universal coefficient theorem for cohomology, $H_*(f^\vee)$ is an isomorphism since $H_*(f)$ is an isomorphism. The proposition follows nows from the square (42). $\square$

Theorem 43. *Let M be a simply-connected oriented Poincaré duality space of formal dimension d. Let G be a topological group such that M is a classifying space for G or let G be ΩM the (Moore) pointed loop space on M. Let $[M] \in H_d(M)$ be its fundamental class. Let c the image of $[M]$ through the composite*

$$H_*(M) \overset{H_*(s)}{\to} H_*(LM) \overset{BFG^{-1}}{\to} HH_*(S_*(G), S_*(G)).$$

Then

a) The morphism of left $HH^*(S_*(G), S_*(G))$-modules

$$\mathbb{D}^{-1} : HH^p(S_*(G), S_*(G)) \overset{\cong}{\to} HH_{d-p}(S_*(G), S_*(G)), \quad a \mapsto a.c,$$

is an isomorphism.

b) The Gerstenhaber algebra $HH^*(S_*(G), S_*(G))$ equipped with the operator $\Delta := -\mathbb{D} \circ B \circ \mathbb{D}^{-1}$ is a Batalin-Vilkovisky algebra.

Here s denotes $s : M \hookrightarrow LM$ the inclusion of the constant loops into LM and BFG is the isomorphism of graded $\Bbbk$-modules between the free loop space homology of M and the Hochschild homology of $S_*(G)$ introduced by Burghelea, Fiedorowicz [5] and Goodwillie [19]. Finally B denotes Connes boundary on $HH_*(S_*(G), S_*(G))$.

Remark 44. We expect that the above theorem can be extended to any path-connected topological monoid G instead of just the topological monoid of pointed Moore loop spaces ΩM or instead of just any topological group.

Proof. By [10, Proposition 6.13 in the case F=pt] when G is a topological group or by [10, Theorem 6.3] when $G = \Omega M$, there exists a differential graded coalgebra $B(S_*(EG); S_*(G); \Bbbk)$ and two quasi-isomorphisms of coalgebras

$$B(S_*(G)) \overset{\simeq}{\leftarrow} B(S_*(EG); S_*(G); \Bbbk) \overset{\simeq}{\rightarrow} S_*(M).$$

The induced isomorphism in homology is the well known isomorphism due to Moore [31, Corollary 7.29]

$$\theta : \mathrm{Tor}^{S_*(G)}(\Bbbk, \Bbbk) = H_*(B(S_*(G))) \overset{\cong}{\rightarrow} H_*(M).$$

Let $[m] \in H_*(B(S_*(G)))$ such that $\theta([m]) = [M]$. By Proposition 41 and Example 40, the cap product with $[m]$, $[m] \cap - : B(S_*(G))^\vee \overset{\simeq}{\rightarrow} B(S_*(G))$, $a \mapsto [m] \cap a$ is quasi-isomorphism.

Denote by $ev : LM \twoheadrightarrow M$, $l \mapsto l(0)$ the evaluation map. The isomorphism BFG of Goodwillie, Burghelea and Fiedorowicz fits into the commutative square.

$$\begin{tikzcd}
HH_*(S_*(G), S_*(G)) \arrow[r, "BFG", "\cong"'] \arrow[d, "HH_*(S_*(G),\varepsilon)"'] & H_*(LM) \arrow[d, "H_*(ev)"] \\
HH_*(S_*(G), \Bbbk) \arrow[r, "\theta", "\cong"'] & H_*(M)
\end{tikzcd}$$

Here ε denote the augmentation of $S_*(G)$. Let $c := BFG^{-1} \circ H_d(s)([M])$. Since s is a section of the evaluation map ev, $HH_*(S_*(G), \varepsilon)(c) = [m]$. So the hypotheses of statement 9 are satisfied for $A = S_*(G)$.

Let N be any non-negatively graded $S_*(G)$-bimodule. Since M is simply connected, by Corollary 13, we obtain that the morphism

$$\mathcal{D}^{-1} : HH^p(S_*(G), N) \overset{\cong}{\rightarrow} HH_{d-p}(S_*(G), N), \quad a \mapsto c \cap a$$

is an isomorphism. By taking $N = S_*(G)$ and by passing from a right action to a left action by (30), we obtain a).

The isomorphism BFG of Goodwillie, Burghelea and Fiedorowicz satisfies $\Delta \circ BFG = BFG \circ B$. Consider M equipped with the trivial S^1-action. The section $s : M \hookrightarrow LM$ is S^1-equivariant. Since

$$B(c) = B \circ BFG^{-1} \circ H_d(s)([M]) = BFG^{-1} \circ \Delta \circ H_d(s)([M]) = 0,$$

by Proposition 32, we obtain b). □

8. Proof of the main theorem for discrete groups

Theorem 45. *Let G be a discrete group such that its classifying space $K(G,1)$ is an oriented Poincaré duality space of formal dimension d. Let $[M] \in H_d(G, \Bbbk)$ be a fundamental class. Let c be the image of $[M]$ by $Tor_*^E(\eta,\eta) : H_*(G,\Bbbk) \to HH_*(\Bbbk[G], \Bbbk[G])$ (Property 19 ii)). Then*

a) The morphism of left $HH^*(\Bbbk[G], \Bbbk[G])$-modules

$$\mathbb{D}^{-1} : HH^p(\Bbbk[G], \Bbbk[G]) \overset{\cong}{\to} HH_{d-p}(\Bbbk[G], \Bbbk[G]), a \mapsto a.c$$

is an isomorphism.

b) The Gerstenhaber algebra $HH^*(\Bbbk[G], \Bbbk[G])$ equipped with the operator $\Delta := -\mathbb{D} \circ B \circ \mathbb{D}^{-1}$ is a Batalin-Vilkovisky algebra.

Proof. Let N be any ungraded $\Bbbk[G]$-bimodule. Since, by hypothesis, G is orientable Poincaré duality group, the cap product with $[M]$ in group (co)homology gives an isomorphism ([4, 10.1 iv), Remark 1 and Example 1 p. 222], [16, Th 15.3.1])

$$[M] \cap - : H^p(G, \tilde{N}) \overset{\cong}{\to} H_{d-p}(G, \tilde{N}), a \mapsto [M] \cap a.$$

Therefore, by Corollary 20, the cap product with $c = \sigma([M])$ in Hochschild (co)homology gives the isomorphism

$$c \cap - : HH^p(\Bbbk[G], N) \to HH_{d-p}(\Bbbk[G], N), a \mapsto c \cap a.$$

Taking $N = \Bbbk[G]$ and passing from a right action to left action as in (30), we obtain a).

By i) of Property 19, $\sigma : H_*(G; \Bbbk) \to HH_*(\Bbbk[G], \Bbbk[G])$ commute with Connes boundary map B on $H_*(G; \Bbbk)$ and on $HH_*(\Bbbk[G], \Bbbk[G])$. By a well known result of Karoubi (for example [27, E.7.4.8] or [41, Theorem 9.7.1]), Connes boundary map B is trivial on $H_*(G; \Bbbk)$. Therefore $B(c) = B \circ \sigma([M]) = \sigma \circ B([M]) = 0$. By applying Proposition 32, we obtain b). □

Property 46. Let A and B be two algebras (differential graded if we want). Let N be an $(A, A \otimes B)$-bimodule. Let $c \in HH_d(A, A)$. Then

i) $HH^*(A, N)$ and $HH_*(A, N)$ are two right B-modules and

ii) the cap product

$$c \cap - : HH^p(A, N) \to HH_{d-p}(A, N), \quad a \mapsto c \cap a$$

is a morphism of right B-modules.

Proof. Since N is an (A^e, B)-bimodule, $\mathcal{C}^*(A,N) \cong \mathrm{Hom}_{A^e}(B(A;A;A),N)$ is a (differential graded) right B-module and its homology $HH^*(A,N)$ is a right B-module. Similarly $\mathcal{C}_*(A,N) \cong N \otimes_{A^e} B(A;A;A)$ and $HH_*(A,N)$ are two right B-modules. Let c be $a[a_1|\dots|a_n] \in \mathcal{C}_n(A,A)$. Let $f \in \mathcal{C}^p(A,N)$. By definition, $c \cap f := \pm af([a_1|\dots|a_p])[a_{p+1}|\dots|a_n]$. Therefore for any $b \in B$,

$$(c\cap f)\cdot b = \pm af([a_1|\dots|a_p])b[a_{p+1}|\dots|a_n] = \\ \pm\, a(f\cdot b)([a_1|\dots|a_p])[a_{p+1}|\dots|a_n] = c \cap (f\cdot b).$$

□

Remark 47. We will be only interested in the case $N = A\otimes A$ and $B = A^e$. Here the A-bimodule structure on N is given by $a\cdot(x\otimes y)\cdot b = ax\otimes yb$ and is called the *outer* structure [18, (1.5.1)]. And the right B-module on N is given by $(x\otimes y)\cdot(a\otimes b) = xa\otimes by$, $x\otimes y \in N$, $a\otimes b \in B$ and is called the *inner* structure.

Definition 48. ([18, Definition 3.2.3, (3.2.5), Remark 3.2.8] or simply [2, Definition 2.1]) An ungraded algebra A is *Calabi-Yau* of dimension d if

i) viewed as an A-bimodule over itself, A admits a finite resolution by finite type projective A-bimodules, i. e. there exists an exact sequence of A^e-projective finite type module of the form

$$0 \to P_i \to P_{i-1} \to \cdots \to P_1 \to P_0 \to A \to 0,$$

ii) for all $k \neq d$, $HH^k(A, A\otimes A) = 0$ and

iii) as (A,A)-bimodule, $HH^d(A, A\otimes A)$ is isomorphic to A (Here the (A,A)-bimodule on $HH^*(A, A\otimes A)$ is given by Property 46 and Remark 47).

Proposition 49. *(Stated without proof in* [18, Remark 3.4.2]*) Let A be an ungraded algebra. Let $c \in HH_d(A,A)$. Suppose that for every A-bimodule N, $c\cap - : HH^p(A,N) \xrightarrow{\cong} HH_{d-p}(A,N)$, $a \mapsto c\cap a$, is an isomorphism. Then A satisfies conditions ii) and iii) of Definition 48.*

Proof. Let N be a free (A,A)-bimodule. Then $HH_*(A,N) = 0$ if $* \neq 0$. Therefore $HH^k(A,N) = 0$ if $k \neq d$. Suppose moreover that N is a $(A, A\otimes B)$-bimodule. The quasi-isomorphism of complexes $\mathcal{C}_*(A,N) \cong N\otimes_{A^e} B(A;A;A) \xrightarrow{\simeq} N \otimes_{A^e} A$ is a morphism of right B-modules. By Property 46,

$$c\cap - : HH^d(A,N) \to HH_0(A,N) \cong N\otimes_{A^e} A$$

is an isomorphism of right B-modules.

Let N be the (A,A)-bimodule $A\otimes A$ with the outer structure and $B=A^e$ (See Remark 47). Then $N\otimes_{A^e} A=(A\otimes A)\otimes_{A^e} A\overset{\cong}{\to} A$, $(x\otimes y)\otimes_{A^e} m\mapsto ymx$ is an isomorphism whose inverse is the map mapping $a\mapsto(1\otimes 1)\otimes_{A^e} a$. A straightforward calculation shows that theses isomorphisms are right A^e-linear. Therefore, we have proved that $HH^d(A,A\otimes A)$ is isomorphic to A as right A^e-modules. □

Theorem 50. *Let $\Bbbk$ be any commutative ring. Let G be a orientable Poincaré duality group of dimension d. Then its group ring $\Bbbk[G]$ is a Calabi-Yau algebra of dimension d.*

When $\Bbbk$ is a field of characteristic 0 or of characteristic prime to the cardinal of G, this theorem was proved by Kontsevich [18, Corollary 6.1.4] and Lambre [26, Lemme 6.2].

Proof. By [4, Remark 2. p. 222], there exists a finite resolution $P\overset{\simeq}{\to}\Bbbk$ of $\Bbbk$ by finite type projective $\Bbbk[G]$-left modules. Then $X:=\Bbbk[G\times G^{op}]\otimes_{\Bbbk[G]} P\overset{\simeq}{\to}\Bbbk[G]$ is a finite type resolution of $\Bbbk[G]$ by finite type projective $\Bbbk[G]$-bimodules.

In the proof of Theorem 45,we saw that for any $\Bbbk[G]$-bimodule N, $c\cap - : HH^p(\Bbbk[G],N)\overset{\cong}{\to} HH_{d-p}(\Bbbk[G],N)$, $a\mapsto c\cap a$, is an isomorphism. Therefore, by Proposition 49, $\Bbbk[G]$ is a Calabi-Yau algebra of dimension d. □

9. String topology of classifying spaces

In [7], Chataur and the author, and in [1], Behrend, Ginot, Noohi and Xu developped a string topology theory dual to Chas-Sullivan string topology.

Theorem 51. [1, 7] *Let G be a path-connected compact Lie group of dimension d. Denote by BG its classifying space. Then the shifted free loop space cohomology $H^{*+d}(LBG)$ is a (possibly non-unital) Batalin-Vilkovisky algebra.*

The goal of this section is to prove the following theorem:

Theorem 52. *Let G be a path-connected compact Lie group of dimension d. Denote by $S^*(BG)$ the singular cochains on the classifying space of G. Then*

a) There exists an explicit isomorphism of left $HH^*(S^*(BG),S^*(BG))$-modules

$$\mathbb{D}^{-1} : HH^p(S^*(BG),S^*(BG))\overset{\cong}{\to} HH_{-d-p}(S^*(BG),S^*(BG)).$$

b) The Gerstenhaber algebra $HH^*(S^*(BG), S^*(BG))$ equipped with the operator $\Delta := -\mathbb{D} \circ B \circ \mathbb{D}^{-1}$ is a Batalin-Vilkovisky algebra.

Both Batalin-Vilkovisky algebras in Theorems 51 and 52 are determined by an orientation class of $H_d(G)$. In [23], Jones gave an isomorphism of graded vector spaces

$$J : HH_*(S^*(BG), S^*(BG)) \stackrel{\cong}{\to} H^*(LBG).$$

Again, we conjecture that the isomorphism of graded vector spaces $J \circ \mathbb{D}^{-1} : HH^*(S^*(BG), S^*(BG)) \stackrel{\cong}{\to} H^{*+d}(LBG)$ is a morphism of Batalin-Vilkovisky algebras.

Theorem 52 is the Eckmann-Hilton or Koszul dual of the following theorem proved by Chataur and the author.

Theorem 53. [7, Theorem 54] *Let G be a path-connected compact Lie group of dimension d. Denote by $S_*(G)$ the algebra of singular chains of G. Consider Connes coboundary map $H(B^\vee)$ on the Hochschild cohomology of $S_*(G)$ with coefficients in its dual, $HH^*(S_*(G); S^*(G))$. then there is an isomorphism of graded vector spaces of upper degree d*

$$\mathcal{D}^{-1} : HH^p(S_*(G); S_*(G)) \stackrel{\cong}{\to} HH^{p+d}(S_*(G); S^*(G))$$

such that the Gerstenhaber algebra $HH^*(S_*(G); S_*(G))$ equipped with the operator $\Delta = \mathcal{D} \circ H(B^\vee) \circ \mathcal{D}^{-1}$ is a Batalin-Vilkovisky algebra.

9.1. **Frobenius algebras.** .

Definition 54. Let A be a differential graded algebra. We say that A is a *Frobenius algebra* if there is a quasi-isomorphism of right A-modules $A \stackrel{\simeq}{\to} A^\vee$. In particular, a graded algebra A is a Frobenius algebra if A is isomorphic as right A-modules to its dual $A^\vee$.

Property 55. [29, Theorem 9.8] Let A be a differential graded algebra. Then A is a Frobenius algebra if and only if its homology $H(A)$ is a Frobenius algebra.

Proof. Let M be any left A-module. A straightforward computation shows that the linear map $\mu : H(\mathrm{Hom}(M, \Bbbk)) \to \mathrm{Hom}(H(M), \Bbbk)$ mapping a cycle $f : M \to \Bbbk$ to $H(f) : H(M) \to \Bbbk$ is a morphism of right $H(A)$-modules. Since in this section, $\Bbbk$ is a field, by the universal coefficient theorem for cohomology, this map μ is an isomorphism. We are only interested in the case $M = A$.

Suppose that we have an quasi-isomorphism of right A-modules $\Theta : A \stackrel{\simeq}{\to} A^\vee$. Then the composite $H(A) \stackrel{H(\Theta)}{\to} H(A^\vee) \stackrel{\mu}{\to} H(A)^\vee$ is an isomorphism of right $H(A)$-modules.

Conversely, suppose that we have an isomorphism of right $H(A)$-modules, $\Theta : H(A) \overset{\cong}{\to} H(A)^\vee$. Then the composite $H(A) \overset{\Theta}{\to} H(A)^\vee \overset{\mu^{-1}}{\to} H(A^\vee)$ is also an isomorphism of right $H(A)$-modules. Let x be a cycle of $A^\vee$ such that $\mu^{-1} \circ \Theta(1) = [x]$.The morphism of right A-modules $A \to A^\vee$, $a \mapsto xa$, coincides in homology with the isomorphism $\mu^{-1} \circ \Theta$. $\square$

Corollary 56. *Let A and B be two differential graded algebras such that $H(A) \cong H(B)$ as graded algebras. Then A is Frobenius if and only if B is.*

Note that it is not necessary that there is a quasi-isomorphism of algebras $f : A \overset{\simeq}{\to} B$ (Compare with Proposition 41 or [29, Corollary 9.9]).

Property 57. Let A be a graded algebra and let C be a graded coalgebra. Consider a bilinear form $< , >: C \otimes A :\to \mathbb{k}$. Let $\phi : A \to C^\vee$, defined by $\phi(a)(c) = (-1)^{|a||c|} < c, a >$, and let $\psi : C \to A^\vee$, defined by $\psi(c)(a) =< c, a >$, be the right and left adjoints. Suppose that ϕ is a morphism of graded algebras. Then

i) ψ is a morphism of right A-modules with respect to the cap product (38) associated to the coalgebra C , i. e. $\psi(c \cap \phi(a)) = \psi(c).a$ for any $c \in C$ and $a \in A$.

ii) If A is non-negatively graded and of finite type in each degree then $\psi : C \to A^\vee$ is a morphism of graded coalgebras.

Proof. i) Let $\Delta c = \sum c' \otimes c"$ be the diagonal of c. By definition, the cap product $c \cap \phi(a)$ is equal to $\sum (-1)^{|c"||a|} < c', a > c"$. Therefore $\psi(c \cap \phi(a))$ is the form on A, mapping $x \in A$ to $\sum (-1)^{|c"||a|} < c', a >< c", x >$. On the other hand, $\psi(c).a$ is the form on A mapping $x \in A$ to $< c, ax >$. But ϕ is a morphism of algebras if only and if for every a, $x \in A$ and $c \in C$, $< c, ax >= \sum (-1)^{|c"||a|} < c', a >< c", x >$. $\square$

Let us give a well-known application of i) of Property 57. Let $C = S_*(M)$ and $A = C^\vee = S^*(M)$. We obtain that the quasi-isomorphism $\psi : S_*(M) \to S^*(M)^\vee$ is a morphism of $S^*(M)$-modules [13, Section 7]. Therefore by Poincaré duality, $S^*(M)$ is a Frobenius algebra. And $H^*(M)$ also.

9.2. **String topology of manifolds.** Let M be a closed oriented d-dimensional smooth manifold. Denote by $\mathbb{H}_*(M) := H_{*+d}(M)$. Poincaré duality [21, Theorem 3.30] gives an isomorphism of graded algebras

$$H^*(M) \cong \mathbb{H}_*(M)$$

where

-the product on $H^*(M)$ is the cup product $H^*(\Delta)$,
-the product on $\mathbb{H}_*(M)$ is the intersection product $\Delta_!$ and
-the fundamental class $[M] \in H_d(M)$ is the unit of $\mathbb{H}_*(M)$.

Chas and Sullivan have defined a Batalin-Vilkovisky algebra on $\mathbb{H}_*(LM) := H_{*+d}(LM)$. The Chas-Sullivan loop product on $\mathbb{H}_*(LM)$ mixes the intersection product $\Delta_!$ on $\mathbb{H}_*(M)$ and the Pontryagin product $H_*(comp)$ on $H_*(\Omega M)$.

More precisely, let $\tilde{\Delta} : M^{S^1 \vee S^1} \hookrightarrow LM \times LM$ be the inclusion map and let *comp* : $M^{S^1 \vee S^1} \to LM$ be the map obtained by composing loops. The Chas-Sullivan loop product is the composite

$$H_*(LM \times LM) \stackrel{\tilde{\Delta}_!}{\to} H_{*-d}(M^{S^1 \vee S^1}) \stackrel{H_*(comp)}{\to} H_{*-d}(LM).$$

The loop product admits $H_d(s)([M])$ as unit. More generally $H_*(s) : \mathbb{H}_*(M) \to \mathbb{H}_*(LM)$ is a morphism of algebras preserving the units. Let $i : \Omega M \hookrightarrow LM$ be the inclusion of the pointed loops into the free loops. The shriek map of i, called the intersection map, $i_! : \mathbb{H}_*(LM) \to H_*(\Omega M)$, is also a morphism of algebras preserving the units [6, Proposition 3.4].

The unit of the Batalin-Vilkovisky algebra $\mathbb{H}_*(LM)$ and the fact that $\Delta 1 = 0$ in any unital Batalin-Vilkovisky algebras was the key for proving Theorem 43.

9.3. **Versus string topology of classifying spaces.** . Let G be a path-connected Lie group of dimension d. Denote by $\mathbb{H}^*(\Omega BG) = H^{*+d}(\Omega BG)$. Since $H_*(\Omega BG)$ is a finite dimensional Hopf algebra, $H_*(\Omega BG)$ is a Frobenius algebra: there is an isomorphism of right $H_*(\Omega BG)$-modules [7, Section 4.1]

$$\Theta : H_*(\Omega BG) \cong \mathbb{H}^*(\Omega BG).$$

By [37, Theorem 5.1.2, with left Hopf modules instead of right Hopf modules], the composite of the antipode of the Hopf algebra $H_*(\Omega BG)$ and of Θ, $H_*(\Omega BG) \stackrel{S}{\to} H_*(\Omega BG) \stackrel{\Theta}{\to} H^*(\Omega BG)$ is an isomorphism of left Hopf modules over $H_*(\Omega BG)$, and so coincides with Poincaré duality.

Therefore this isomorphism Θ is an isomorphism of algebras if
-the product on $H_*(\Omega BG)$ is the Pontryagin product $H_*(comp)$,
-the product on $\mathbb{H}^*(\Omega BG)$ is the composite

$$H^*(\Omega BG) \otimes H^*(\Omega BG) \stackrel{\tau}{\to} H^*(\Omega BG) \otimes H^*(\Omega BG) \stackrel{comp^!}{\to} H^{*-d}(\Omega BG)$$

where τ denote the twist map given by $a \otimes b \mapsto (-1)^{|a||b|} b \otimes a$ and $comp^!$ is the shriek map of *comp*.

Of course, $\Theta(1)$ is the unit of the algebra $\mathbb{H}^*(\Omega BG)$.

The product on $\mathbb{H}^*(LBG) := H^{*+d}(LBG)$ mixes the cup product $H^*(\Delta)$ on $H^*(BG)$ and the product $comp^!$ on $\mathbb{H}^*(\Omega BG)$. More precisely, the product on $\mathbb{H}^*(LBG)$ is the composite

$$H^*(LBG \times LBG) \overset{H^*(\tilde{\Delta})}{\to} H^*(BG^{S^1\vee S^1}) \overset{comp^!}{\to} H^{*-d}(LBG).$$

Comparing with the definition of the Chas-Sullivan loop product defined above, we see a general principle. In order to pass from string topology of manifolds to string topology of classifying spaces, you replace

-homology by cohomology,

-shriek map in homology like $\tilde{\Delta}_!$ by the map induced in singular cohomology like $H^*(\tilde{\Delta})$,

-maps induced in singular homology like $H_*(comp)$ by shriek map in cohomology like $comp^!$.

In particular, you never change the direction of arrows.

Guided by this general principle, we now transpose the proof of Theorem 43 into a proof of Theorem 52. Using this general principle, the product on $\mathbb{H}^*(LBG)$ should have $s^!(1)$ as an unit. More generally $s^! : H^*(BG) \to \mathbb{H}^*(LBG)$ should be a morphism of algebras preserving the units. Also $H^*(i) : \mathbb{H}^*(LBG) \to \mathbb{H}^*(\Omega BG)$ should be a morphism of algebras preserving the units. The problem is that $s^!$ is not easy to define [7, Remark 56] and that we have not yet proved the previous assertions. Instead, we are going only to prove the following lemma.

Lemma 58. *There exists an explicit element $\mathbb{I} \in H^d(LBG)$ such that $\Delta\mathbb{I} = 0$ and such that the morphism of right $H_*(\Omega BG)$-modules, $\Theta : H_p(\Omega BG) \overset{\cong}{\to} H^{d-p}(\Omega BG)$, $a \mapsto H^d(i)(\mathbb{I}).a$ is an isomorphism.*

As explained above, we believe that $\mathbb{I}$ is the unit of the Batalin-Vilkovisky algebra $\mathbb{H}^*(LBG)$.

Proof. Let $\eta : \{e\} \to G$ be the unit of G. Consider $\eta_! : H_d(G) \to \mathbb{k}$ the shriek map of η. By Lemma 55 of [7], the morphism of right $H_*(G)$-modules $H_p(G) \overset{\cong}{\to} H^{d-p}(G)$, $a \mapsto \eta_!.a$, is an isomorphism. Consider the commutative diagram of graded algebras

$$\begin{tikzcd}
H^*(LBG) \arrow[r, "H^*(\gamma)", "\cong"'] \arrow[d, "H^*(i)"'] & H^*(|\Gamma G|) \arrow[d, "H^*(|j|)"'] & H^*(EG\times_G G^{ad}) \arrow[l, "H^*(|\Phi)|"', "\cong"] \arrow[dl, "H^*(E\eta\times_\eta G^{ad})"] \\
H^*(\Omega BG) \arrow[r, "H^*(\bar{\gamma})", "\cong"'] & H^*(G) &
\end{tikzcd}$$

where the right triangle is the triangle considered in the proof of Theorem 54 of [7] and the left square is induced by the following commutative square of topological spaces

$$\begin{tikzcd} G \arrow[r, "|j|"] \arrow[d, "\bar{\gamma}", "\simeq"'] & {|\Gamma G|} \arrow[d, "\gamma", "\simeq"'] \\ \Omega BG \arrow[r, "i"] & LBG \end{tikzcd}$$

considered in the proof of Theorem 7.3.11 of [27]. Consider the equivariant Gysin map: $EG \times_G \eta^! : H^*(BG) \to H^{*+d}(EG \times_G G^{ad})$. Let $\mathbb{I}$ be the image of 1 by the composite $H^*(\gamma)^{-1} \circ H^*(|\Phi|) \circ EG \times_G \eta^!$. In [7, (58)], we saw that $\Delta\mathbb{I} = 0$. By Lemma 57 of [7], $H^*(E\eta \times_\eta G^{ad})$ maps $EG \times_G \eta^!(1)$ to $\eta_! \in H_d(G)^\vee$. Therefore using the above commutative diagram, $H^*(i)(\mathbb{I}) = H^*(\bar{\gamma})^{-1}(\eta_!)$.

By Lemma 7.3.12 of [27], $\bar{\gamma} : G \overset{\simeq}{\to} \Omega BG$ is the classical homotopy equivalence which is well-known to be a morphism of H-spaces. Therefore the isomorphism induced in homology, $H_*(\bar{\gamma}) : H_*(G) \overset{\cong}{\to} H_*(\Omega BG)$, is a morphism of algebras. Since $H_*(G)$ is a Frobenius algebra, $H_*(\Omega BG)$ is also a Frobenius algebra. More precisely, the morphism of right $H_*(\Omega BG)$-modules $\Theta : H_p(\Omega BG) \to H_{d-p}(\Omega BG)^\vee$, $a \mapsto H^*(\bar{\gamma})^{-1}(\eta_!).a$ is an isomorphism. □

To finish the proof of Theorem 52, we need also the following algebraic results.

9.4. **Bar and Cobar construction.** Let C be a coaugmented DGC. Denote by $\overline{C}$ the kernel of the counit. The normalized *cobar construction on C*, denoted ΩC, is the augmented differential graded algebra $\left(T(s^{-1}\overline{C}), d_1 + d_2\right)$ where d_1 and d_2 are the unique derivations determined by

$$d_1 s^{-1} c = -s^{-1} dc \text{ and}$$
$$d_2 s^{-1} c = \sum_i (-1)^{|x_i|} s^{-1} x_i \otimes s^{-1} y_i,\ c \in \overline{C}$$

where the reduced diagonal $\overline{\Delta} c = \sum_i x_i \otimes y_i$. We follow the sign convention of [9].

Remark 59. [20, (A.6)] A bilinear form $< , >: V \otimes W \to \Bbbk$ of graded vector spaces extends a bilinear form $< , >: TV \otimes TW \to \Bbbk$ defined by

$$< v_1 \otimes \cdots \otimes v_i, w_1 \otimes \cdots \otimes w_i >= \pm \prod_{j=1}^{i} < v_j, w_j >$$

and $< v_1 \otimes \cdots \otimes v_i, w_1 \otimes \cdots \otimes w_j >= 0$ if $i \neq j$. Here again $\pm$ is the sign given by the Koszul sign convention.

Proposition 60. *Let C be a coaugmented differential graded coalgebra. Denote by $A := C^\vee$ the differential graded algebra dual of C. Let $<\ ,\ >$: $sA \otimes s^{-1}C \to \mathbb{k}$ be the non-degenerate bilinear form defined in* [20, p. 276 in the case $V = s^{-1}C$ and $X = A$] *by* $< sa, s^{-1}c >= (-1)^{|a|+1}a(c)$. *Consider the bilinear form $<\ ,\ >$: $BA \otimes \Omega C \to \mathbb{k}$ extending $<\ ,\ >$: $sA \otimes s^{-1}C \to \mathbb{k}$ (Remark 59). Then*

i) the right adjoint $\phi : \Omega C \to (BA)^\vee$ is a natural morphism of differential graded algebras and the left adjoint $\psi : BA \to (\Omega C)^\vee$ is a natural morphism of complexes,

ii) if C is of finite type in each degre and $C = \mathbb{k} \oplus C_{\geq 2}$ then both ϕ and ψ are isomorphisms,

iii) if $H(C)$ is of finite type in each degre and $C = \mathbb{k} \oplus C_{\geq 2}$ then both $H(\phi)$ and $H(\psi)$ are isomorphisms.

Proof. i) and ii) Denote by TAW the tensor algebra on W, and by TCV the tensor coalgebra on V [20, p. 277-8]. It is easy to check that the right adjoint map $\phi : TAW \to TCV^\vee$ of the bilinear map defined by Remark 59 is a morphism of graded algebras. In [32, Proof of Theorem 6.1 ii)], we have checked carefully that $\psi : \mathcal{C}_*(A, A) \to (C \otimes \Omega C, \delta)^\vee$, where $(C \otimes \Omega C, \delta)$ is the cyclic cobar complex of C, is a morphism of complexes and an isomorphism if C is of finite type in each degree and $\overline{C} = \overline{C}_{\geq 2}$. The same proof shows that this is also the case for $\psi : BA \to \Omega C^\vee$.

iii) By Proposition 4.2 of [9], there exists a differential graded algebra of the form (TV, d) where $V = V^{\geq 2}$ is of finite type in each degree and a quasi-isomorphism of augmented differential graded algebras $f : TV \overset{\simeq}{\to} C^\vee$. By ii) of Property 57, the adjoint map $g : C \overset{\cong}{\to} (C^\vee)^\vee \underset{\simeq}{\overset{f^\vee}{\to}} TV^\vee$ is a quasi-isomorphism of coaugmented differential graded coalgebras [12, p. 56]. Denote by $D := TV^\vee$.

Since $\overline{C}_{\leq 1} = \overline{D}_{\leq 1} = 0$, by Remark 2.3 of [9], $\Omega f : \Omega C \overset{\simeq}{\to} \Omega D$ is a quasi-isomorphism of augmented differential graded algebras. Since $\mathbb{k}$ is a field, $f^\vee : D^\vee \overset{\simeq}{\to} C^\vee$ is also a quasi-isomorphism of augmented differential graded algebras. By naturality of ψ, we have the commutative square of complexes
$$\begin{tikzcd} B(C^\vee) \arrow[r, "\psi"] & (\Omega C)^\vee \\ B(D^\vee) \arrow[u, "B(f^\vee)", "\simeq"'] \arrow[r, "\psi", "\cong"'] & (\Omega D)^\vee \arrow[u, "\simeq", "(\Omega f)^\vee"'] \end{tikzcd}$$
where the two vertical morphisms are quasi-isomorphisms. By ii), $\psi : B(D^\vee) \overset{\cong}{\to}$

$(\Omega D)^\vee$ is an isomorphism. Therefore $\psi : B(C^\vee) \overset{\simeq}{\to} (\Omega C)^\vee$ is a quasi-isomorphism. Similarly, one proves that $\phi : \Omega C \overset{\simeq}{\to} B(C^\vee)^\vee$ is also a quasi-isomorphism. □

Proof of Theorem 52. The Eilenberg Moore formula gives an isomorphism of graded algebras $\mathcal{EM} : H_*(\Omega BG) \overset{\cong}{\to} H(\Omega S_*(BG))$. By Proposition 60 iii), $\psi : BS^*(BG) \overset{\simeq}{\to} \Omega S_*(BG)^\vee$ is a quasi-isomorphism of complexes. The Jones isomorphism J fits into the commutative diagram

$$\begin{tikzcd} HH_*(S^*(BG), S^*(BG)) \arrow[rr, "J", "\cong"'] \arrow[d, "HH_*(S^*(BG),\varepsilon)"'] & & H^*(LBG) \arrow[d, "H^*(i)"] \\ \mathrm{Tor}^{S^*(BG)}(\Bbbk, \Bbbk) \arrow[r, "H(\psi)", "\cong"'] & H(\Omega S_*(BG))^\vee \arrow[r, "\mathcal{EM}^\vee", "\cong"'] & H^*(\Omega BG) \end{tikzcd}$$

Consider the element $\mathbb{I} \in H^d(LBG)$ given by Lemma 58. Let c be $J^{-1}(\mathbb{I}) \in HH_{-d}(S^*(BG), S^*(BG))$. Denote by $m \in BS^*(BG)$ a cycle such that its class $[m]$ is equal to $HH_{-d}(S^*(BG), \varepsilon)(c)$.

Since $H_*(\Omega BG)$ is a Frobenius algebra, $H(\Omega S_*(BG))$ is also a Frobenius algebra. More precisely, by Lemma 58, the morphism of right $H_*(\Omega BG)$-modules $H_p(\Omega BG) \overset{\cong}{\to} H_{d-p}(\Omega BG)^\vee$ mapping 1 to $H^d(i)(\mathbb{I})$ is an isomorphism. Therefore the morphism of right $H(\Omega S_*(BG))$-modules $H_p(\Omega S_*(BG)) \overset{\cong}{\to} H_{d-p}(\Omega S_*(BG))^\vee$ mapping 1 to $(\mathcal{EM}^\vee)^{-1} \circ H^d(i)(\mathbb{I})$ is an isomorphism. Since the above diagram is commutative, $(\mathcal{EM}^\vee)^{-1} \circ H^d(i)(\mathbb{I}) = H(\psi)([m])$. By Property 55, the differential graded algebra $\Omega S_*(BG)$ is a Frobenius algebra. More precisely, the morphism of right $\Omega S_*(BG)$-modules $\theta : \Omega S_*(BG) \overset{\simeq}{\to} (\Omega S_*(BG))^\vee$, $a \mapsto \psi(m).a$ is a quasi-isomorphism.

By Proposition 60, $\phi : \Omega S_*(BG) \overset{\simeq}{\to} BS^*(BG)^\vee$ is a quasi-isomorphism of differential graded algebras. Therefore by i) of Property 57, the following square of complexes commutes.

$$\begin{tikzcd} \Omega S_*(BG) \arrow[r, "\phi", "\simeq"'] \arrow[d, "\theta"', "\simeq"] & (BS^*(BG))^\vee \arrow[d, "\simeq"', "m\cap -"] \\ (\Omega S_*(BG))^\vee & BS^*(BG) \arrow[l, "\simeq", "\psi"'] \end{tikzcd}$$

Therefore (Example 40),

$$[m] \cap - : \mathrm{Ext}^p_{S^*(BG)}(\Bbbk, \Bbbk) \overset{\cong}{\to} \mathrm{Tor}^{S^*(BG)}_{-d-p}(\Bbbk, \Bbbk)$$

is an isomorphism.

Let N be any non-negatively upper graded $S^*(BG)$-bimodule. Since BG is path-connected, by Corollary 14, we obtain that the morphism

$$\mathcal{D}^{-1}: HH^p(S^*(BG),N) \overset{\cong}{\to} HH_{-d-p}(S^*(BG),N), \quad a \mapsto c \cap a$$

is an isomorphism. By taking $N = S^*(BG)$ and by passing from a right action to a left action by (30), we obtain a).

The isomorphism J of Jones satisfies $\Delta \circ J = J \circ B$. Since by Lemma 58,

$$B(c) = B \circ J^{-1}(\mathbb{I}) = J^{-1} \circ \Delta(\mathbb{I}) = 0,$$

by Proposition 32, we obtain b).

□

10. Appendix

The key of the proof of Proposition 32 is the relation

$$i_{\{a,b\}} = (-1)^{|a|+1}[[B, i_a], i_b] = [[i_a, B], i_b].$$

In this appendix, we recall that $[[i_a, B], i_b]$ is the *derived bracket* of i_a and i_b and we explain that this relation means that the morphism of graded algebras

$$HH^*(A,A) \to \mathrm{End}(HH_*(A,A)), \quad a \mapsto i_a,$$

is a morphism of generalized Loday-Gerstenhaber algebras (Theorem 67)

Definition 61. [24, p. 1247] A *generalized Loday-Gerstenhaber algebra* is a (not necessarily commutative) graded algebra A equipped with a linear map $\{-,-\} : A_i \otimes A_j \to A_{i+j+1}$ of degree 1 such that:
a) the bracket $\{-,-\}$ gives A a structure of graded Leibniz algebra of degree 1. This means that for each a, b and $c \in A$
$\{a,\{b,c\}\} = \{\{a,b\},c\} + (-1)^{(|a|+1)(|b|+1)}\{b,\{a,c\}\}$.
b) the product and the Leibniz bracket satisfy the following relation called the Poisson relation:

$$\{a, bc\} = \{a,b\}c + (-1)^{(|a|+1)|b|}b\{a,c\}.$$

Proposition 62. *Let A be a graded algebra equipped with an operator $d : A_n \to A_{n+1}$ such that $d \circ d = 0$ and such that d is a derivation. Then A equipped with the derived bracket defined by* [24, (2.8)]

$$[a,b]_d := (-1)^{|a|+1}[da, b]$$

is a generalized Loday-Gerstenhaber algebra.

Proof. Since A is an associative graded algebra, the bracket $[-,-]$ defined by

$$[a,b] := ab - (-1)^{|a||b|}ba,$$

is a Lie bracket. Since d is a derivation for the associative product of A, d is a derivation for the Lie bracket $[-,-]$. Therefore by [24, Proposition 2.1], the derived bracket $[-,-]_d$ satisfies the graded Jacobi identity and d is a derivation for the derived bracket $[-,-]_d$. Since $[-,-]_d$ does not satisfy in general anticommutativity, $[-,-]_d$ is only a Leibniz bracket in the sense of Loday [28], and not a Lie bracket in general. The Lie bracket $[-,-]$ satisfies the Poisson relation:

$$[a,bc] = [a,b]c + (-1)^{(|a|+1)|b|}b[a,c].$$

Therefore since $[a,-]_d$ is the derivation $(-1)^{|a|+1}[da,-]$, the Leibniz bracket $[-,-]_d$ also satisfies the Poisson relation [24, Proposition 2.2]:

$$[a,bc]_d = [a,b]_d c + (-1)^{(|a|+1)|b|}b[a,c]_d.$$

□

Remark 63. In Proposition 62, if instead, we define the bracket by

$$[a,b]_d := ad(b) - (-1)^{(|a|+1)(|b|+1)}bd(a)$$

then $[-,-]_d$ satisfies anti-commutativity and Jacobi: $[-,-]_d$ is a Lie bracket [1] of degree $+1$. But this time, $[-,-]_d$ does not satisfy the Poisson relation. Note that again d is a derivation for $[-,-]_d$.

Proof. Let $a \in A_{x-1}$, $b \in B_{y-1}$ and $c \in C_{z-1}$ be three elements of A of degrees $x-1$, $y-1$ and $z-1$. Then

$$\begin{aligned}[a,[b,c]_d]_d = {} & ad(bdc) - (-1)^{zy}ad(cdb)\\ & - (-1)^{xy+xz}b(dc)(da) + (-1)^{xy+xz+yz}c(db)(da),\end{aligned}$$

$$\begin{aligned}[[a,b]_d,c]_d = {} & a(db)(dc) - (-1)^{xy}b(da)(dc)\\ & - (-1)^{zx+zy}cd(adb) + (-1)^{zx+zy+xy}cd(bda)\end{aligned}$$

and

$$\begin{aligned}(-1)^{xy}[b,[a,c]_d]_d = {} & (-1)^{xy}bd(adc) - (-1)^{xy+xz}bd(cda)\\ & - (-1)^{yz}a(dc)(db) + (-1)^{yz+xz}c(da)(db).\end{aligned}$$

Since d is a derivation and $d^2 = 0$, $d(adb) = (da)(db)$. Therefore we have the Jacobi identity:

$$[a,[b,c]_d]_d = [[a,b]_d,c]_d + (-1)^{xy}[b,[a,c]_d]_d.$$

[1] We could not find this Lie bracket in the litterature. So this Lie algebra structure might be new.

Since $[da, b]_d = (da)(db)$ and $[a, db]_d = -(-1)^{x(y+1)}(db)(da)$,

$$d([a, b]_d) = (da)(db) - (-1)^{xy}(db)(da) = [da, b]_d + (-1)^x[a, db]_d.$$

This means that d is a derivation for $[-, -]_d$. □

Example 64. (interior derivation) Let A be an associative graded algebra. Let $\tau \in A_1$ such that $\tau^2 = 0$. Then $d := [\tau, -]$ is a derivation of the associative product and $d \circ d = 0$. Therefore, we can apply the previous proposition. In this case, we denote the derived bracket $[a, b]_d$ simply by $[a, b]_\tau$ and [24, Example p. 1250]

$$[a, b]_\tau = (-1)^{|a|+1}[[\tau, a], b] = [[a, \tau], b].$$

Corollary 65. [24, Beginning of Section 2.4] *Let E be a graded $\mathbb{k}$-module equipped with an operator $B : E_n \to E_{n+1}$ such that $B \circ B = 0$. Then $End(E)$ equipped with the derived bracket $[a, b]_B = [[a, B], b]$ is a generalized Loday-Gerstenhaber algebra.*

Proof. Apply Proposition 62 and Example 64, to $\mathrm{End}(E)$ equipped with the composition product. □

Theorem 66. *(implicit in* [24, p. 1269-70 pointed by Krasilshchik]*) Let A be a Batalin-Vilkovisky algebra. The morphism of graded algebras induced by left multiplication*

$$\Psi : A \to End(A), a \mapsto l_a$$

is an injective morphism of generalized Loday-Gerstenhaber algebras.

Proof. Since A is a graded module equipped with an operator $\Delta : A_n \to A_{n+1}$ such that $\Delta \circ \Delta = 0$, by Corollary 65 applied to A and to $B = -\Delta$, $\mathrm{End}(A)$ equipped with the *derived bracket* $[f, g]_{-\Delta} = [[f, -\Delta], g]$ is a generalized Loday-Gerstenhaber algebra. By Proposition 28,

$$l_{\{a,b\}} = -[[l_a, \Delta], l_b] = [[l_a, B], l_b]$$

Therefore Ψ is a morphism of generalized Loday-Gerstenhaber algebra □

Theorem 67. *Let A be a differential graded algebra. Then*
1) $EndHH_*(A, A)$ equipped with the derived bracket

$$[a, b]_B = [a, B], b]$$

is a generalized Loday-Gerstenhaber algebra.
2) The morphism of graded algebras induced by the action

$$\Phi : HH^*(A, A) \to EndHH_*(A, A), \quad a \mapsto i_a,$$

is a morphism of generalized Loday-Gerstenhaber algebra. In particular, its image $\Phi(HH^*(A, A))$ is a Gerstenhaber algebra.

Proof. Since Connes boundary $B : HH_*(A,A) \to HH_{*+1}(A,A)$ satisfies $B \circ B = 0$, by Corollary 65, we obtain 1).

Since $i_{ab} = i_a \circ i_b$ (equation (31)) and $i_{\{a,b\}} = [[i_a, B], i_b] = [i_a, i_b]_B$, Ψ is a morphism of generalized Gerstenhaber-Loday algebra.

Since $HH^*(A,A)$ is a Gerstenhaber algebra, $\Phi(HH^*(A,A))$ is also a Gerstenhaber algebra. □

Remark 68. If A is a differential graded algebra satisfying the hypotheses of Proposition 32, the morphism $\Phi : HH^*(A,A) \hookrightarrow \mathrm{End} HH_*(A,A)$ of Theorem 67 is injective and can be identified with the morphism Ψ of Theorem 66 for the Batalin-Vilkovisky algebra $HH^*(A,A)$.

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