UNIVERSITÉ DES SCIENCES ET TECHNOLOGIES DE LILLE

SUR L'ALGÈBRE DE COHOMOLOGIE D'UNE FIBRE

Thèse soutenue le 5 Juillet 1997

par

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pour obtenir le titre de

Docteur en Mathématiques

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Numéro d'ordre : 2039

A la mémoire de mon père,
à ma mère.

Remerciements

En premier lieu, je tiens à remercier mon directeur de thèse Nicolas DUPONT pour sa patience, sa disponibilité et ses encouragements constants. C'est lui qui m'a formé à l'homotopie rationnelle et plus généralement à la théorie des algèbres differentielles graduées.

Je dois aussi beaucoup à Stephen HALPERIN. Je le remercie pour les nombreuses heures qu'il m'a consacrées (dans son bureau) à Toronto, pendant l'été 1996.

Kathryn HESS et Jean-Claude THOMAS m'ont fait l'honneur d'être rapporteurs de cette thèse. Les observations de Jean-Claude THOMAS ont eclairci de nombreux points obscurs pour moi.

Je tiens à exprimer ma gratitude à Yves FÉLIX et Daniel TANRÉ, qui m'ont suivi depuis mon DEA. A Louvain-la-Neuve, Yves FÉLIX a toujours été disponible pour répondre à mes questions. Daniel TANRÉ a bien voulu présider mon jury de thèse, qu'il en soit remercié ainsi que tous les membres de mon jury.

J'ai eu de nombreuses discussions profitables avec Octavian CORNEA, Jean-Paul DOERAENE, Pascal LAMBRECHTS et Bitjong N'DOMBOL en particulier sur les suites spectrales, les catégories à modèles et les produits tensoriels tordus.

Ginette DOCLOT, Elisabeth LOCATELLI et Sonia QUAREZ m'ont prodigué une aide précieuse alors que j'étais "perdu" à la bibliothèque ou dans l'administration.

Enfin, je tiens à remercier l'U.F.R. de Mathématiques et l'U.R.A. G.A.T. qui m'ont accueilli ces quatres dernières années à Lille et ont soutenu mon travail de recherche.

Table des matières

I Introduction 4

1 Quelle est la problématique ? 5

2 L'homotopie rationnelle et le théorème du modèle de la fibre 7

3 Le théorème du pseudo-modèle de la fibre 11

II On the cohomology algebra of a fiber 14

4 Introduction 14

5 Algebraic preliminaries and notations 16

6 The bar construction with coefficients as a DGC 20

7 Homotopy of chain complexes and of augmented DGA's 22

8 Bar construction and homotopies 26

9 Bar construction of Hopf algebras up to homotopy 33

10 The fiber of a suspension map 36

11 The category of augmented CDGA's as a category of cofibrant objects 39

12 The fiber of the model in the Anick range 41

III Annexes 48

13 Détails des preuves de 8 et 9 48

14 Preuve du lemme 11.8(iii) 57

Première partie
Introduction

1 Quelle est la problématique ?

Le but de la topologie algébrique est de résoudre des problèmes topologiques. Par exemple, soient X et Y deux espaces topologiques, on peut se demander si X et Y ont le même type d'homotopie : $X \overset{?}{\approx} Y$. On dit que deux applications continues f_0 et f_1 entre espaces topologiques X et Y sont *homotopes* s'il existe une application continue $H : X \times [0,1] \to Y$ telle que pour tout point x de X, $H(x,0) = f_0(x)$ et $H(x,1) = f_1(x)$. On dit que X et Y ont le *même type d'homotopie*, noté $X \approx Y$, s'il existe deux applications continues $g : X \to Y$ et $h : Y \to X$, telles que $g \circ h$ et $h \circ g$ soient respectivement homotopes à l'identité de Y et à l'identité de X. La méthode utilisée en topologie algébrique est la modélisation algébrique. Au lieu de travailler directement sur les espaces topologiques et les applications continues, on préfère modéliser les espaces topologiques par des objets algébriques et les applications continues par des morphismes entre ces objets algébriques.

Généralement, cette modélisation algébrique se fait par un foncteur de la catégorie des espaces topologiques vers une catégorie algébrique. Par exemple, la cohomologie à coefficients dans $\Bbbk$, notée H, est un foncteur contravariant de la catégorie des espaces topologiques vers celle des $\Bbbk$-modules gradués ($\Bbbk$ étant un anneau commutatif) : A tout espace topologique X, on associe la famille de $\Bbbk$-modules $H^*(X)$, $* \in \mathbb{N}$. A toute application continue $f : X \to Y$, on associe la famille de morphismes de $\Bbbk$-modules $H^*(f) : H^*(X) \leftarrow H^*(Y)$. Grâce à un foncteur, on passe d'un problème topologique à un problème algébrique qu'on espère, lui, pouvoir résoudre. Exemple : on sait que si X a le même type d'homotopie que Y, alors $H^*(X)$ est isomorphe à $H^*(Y)$. Ici, on est passé d'une question topologique “deux espaces topologiques ont-ils le même type d'homotopie ? ” à une question d'isomorphisme linéaire.

Le problème est que si on trouve un isomorphisme entre $H^*(X)$ et $H^*(Y)$, on ne peut pas affirmer que X et Y ont le même type d'homotopie. Une solution consiste à enrichir la structure algébrique de notre catégorie algébrique. Dans notre exemple, c'est de voir la cohomologie H comme un foncteur qui arrive non seulement dans la catégorie des modules gradués, mais dans la catégorie, plus petite, des algèbres graduées. En effet, $H^*(X)$ est une algèbre graduée par le cup produit, qui, à isomorphisme près, ne dépend que du type d'homotopie de X.

Malheureusement, il existe encore des espaces X et Y qui n'ont pas le même type d'homotopie : $X \not\approx Y$, mais tels que $H^*(X)$ soit isomorphe à $H^*(Y)$ comme algèbre. Pour pouvoir distinguer des espaces topologiques qui n'ont pas le même type d'homotopie, les topologues algébristes ont continué à considérer des structures algébriques de plus en plus en compliquées. Par exemple, la cohomologie $H^*(X;\mathbb{Z}_p)$ d'un espace topologique X à coefficients dans $\mathbb{Z}_p$ peut être vue comme un module sur une algèbre graduée $\mathcal{A}_p$ appelée algèbre de Steenrod.

Ces structures algébriques sont de plus en plus fines. Mais pour l'instant, elles ne permettent pas encore de dire que deux espaces ont le même type d'homotopie. On se heurte ici à un des problèmes fondamentaux des mathématiques : trouver une modélisation algébrique qui rende compte complètement du problème topologique initial.

Comme nous venons de le voir, la classification homotopique des espaces topologiques passe par des approximations successives de plus en plus fines. Pour résoudre complètement le problème, encore faut il pouvoir calculer explicitement ces approximations successives. En particulier, si on nous donne un espace topologique X, il faut pouvoir calculer sa cohomologie $H^*(X)$.

Souvent un espace topologique intervient dans une fibration.

Definition 1.1 Une application continue $p : E \to B$ est une *fibration* si p a la propriété de relèvement des homotopies : pour tout diagramme commutatif d'espaces topologiques

$$\begin{array}{ccc} X \times 0 & \xrightarrow{f_0} & E \\ \downarrow & \overset{H}{\nearrow} & \downarrow p \\ X \times [0,1] & \xrightarrow[G]{} & B \end{array}$$

il existe une homotopie $H : X \times [0,1] \to E$ qui étende f_0 et relève G. Si B est connexe par arcs, p est surjectif et on appelle *fibre* de p, notée F, $p^{-1}(b)$ où $b \in B$. On démontre facilement que le type d'homotopie de F ne dépend pas du point b choisi.

Les fibrés localement triviaux de base B paracompacte sont des exemples de fibrations que l'on rencontre couramment en géométrie.

Pour toute application continue $f : X \to B$, il existe une fibration $p : E \twoheadrightarrow B$, appelée *fibration associée* à f, telle que f se factorise en une équivalence d'homotopie $X \xrightarrow{\simeq} E$ suivie de p. La fibre de la fibration associée à f est par définition la *fibre homotopique* de f. La fibration associée à $* \to B$ est la fibration des chemins $PB \twoheadrightarrow B$ de fibre l'espace des lacets ΩB.

Soit $F \hookrightarrow E \overset{p}{\twoheadrightarrow} B$ une fibration. Une question fondamentale est de savoir quelles sont les données algébriques sur p qui, à la fois, déterminent et permettent de calculer $H^*(F)$.

L'application continue p induit au niveau des complexes de cochaînes singulières un morphisme d'algèbres différentielles graduées (ADG) $C^*(p) : C^*(B) \to C^*(E)$. La structure de $C^*(B)$-module à droite induite par $C^*(p)$ sur $C^*(E)$ détermine la structure d'espace vectoriel de $H^*(F)$. En effet, d'après Eilenberg-Moore, l'espace vectoriel $H^*(F)$ est isomorphe au produit de torsion $\mathrm{Tor}^{C^*(B)}(C^*(E), \Bbbk)$. Cette formule permet effectivement de calculer $H^*(F)$. En effet, nous pouvons remplacer dans cette formule, l'ADG $C^*(B)$ par une ADG A plus “petite” (i. e. de dimension finie en chaque degré si $H^*(B)$ l'est aussi) et le $C^*(B)$-module $C^*(E)$ par un A-module lui aussi plus petit ([Y. Félix, S. Halperin, J.-C. Thomas, *Adam's cobar equivalence*] 4.2, 4.6 et [J. Mc Cleary, *User's Guide To Spectral Sequences*] 7.6).

Les suites spectrales de Serre et d'Eilenberg-Moore permettent parfois de calculer $H^*(F)$ comme algèbre. Mais en général, sur un corps quelconque, on ne sait toujours pas calculer l'algèbre de cohomologie $H^*(F)$. Voilà la question générale à laquelle j'ai essayé de répondre durant ma thèse. Le résultat le plus marquant que j'ai obtenu est une généralisation d'un théorème classique de l'homotopie rationnelle, appelé le théorème du modèle de la fibre, que je présente maintenant.

2 L'homotopie rationnelle et le théorème du modèle de la fibre

Les topologues algébristes ne savent pas trouver une modélisation algébrique complète du problème $X \overset{?}{\approx} Y$ pour tous les espaces topologiques. Par contre, si on se limite aux espaces topologiques dits rationnels, la théorie de l'homotopie rationnelle fournit une modélisation algébrique complète des types d'homotopies, appelée les modèles de Sullivan.

Definition 2.1 Un espace topologique X est *rationnel* si $H^*(X)$ est un $\mathbb{Q}$-espace vectoriel.

Nous allons présenter les modèles de Sullivan. Le complexe de cochaînes singulières $C^*(X)$ est muni d'une structure de complexe et d'algèbre telle que la différentielle d soit compatible avec la multiplication. En effet, pour tout x, $y \in C^*(X)$,

$$d(xy) = (dx)y + (-1)^{|x|}x(dy)$$

où $|x|$ désigne le degré de x. Cela veut dire que $C^*(X)$ est une algèbre différentielle graduée (ADG). La cohomologie de l'ADG $C^*(X)$, $H^*(X)$ est une algèbre graduée commutative, car pour tout x, $y \in H^*(X)$,

$$xy = (-1)^{|x||y|}yx.$$

Par contre l'ADG $C^*(X)$ n'est pas commutative. Mais, dans le cas des coefficients rationnels, il existe une ADG $D(X)$ et une algèbre différentielle graduée commutative (ADGC) $A(X)$ telles que

$$A(X) \overset{\simeq}{\leftarrow} D(X) \overset{\simeq}{\rightarrow} C^*(X).$$

On désigne ici par $\overset{\simeq}{\rightarrow}$ un morphisme d'ADG qui induit un isomorphisme en homologie. Ces morphismes sont naturels. Par exemple, soit $f : S^2 \hookrightarrow \mathbb{CP}^2$ l'inclusion de CW-complexes, alors on a le diagramme commutatif d'ADG

$$\begin{array}{ccc}
C^*(\mathbb{CP}^2) & \xrightarrow{C^*(f)} & C^*(S^2) \\
\uparrow \simeq & & \uparrow \simeq \\
D(\mathbb{CP}^2) & \longrightarrow & D(S^2) \\
\downarrow \simeq & & \downarrow \simeq \\
A(\mathbb{CP}^2) & \xrightarrow[A(f)]{} & A(S^2)
\end{array}$$

Soient B, C deux algèbres graduées. Alors le produit tensoriel $B \otimes C$ est une algèbre graduée par

$$(b \otimes c)(b' \otimes c') = (-1)^{|c||b'|}bb' \otimes cc'.$$

Soit V un espace vectoriel gradué. L'algèbre graduée commutative libre sur V, notée ΛV, est le produit tensoriel

$$P(V^{pair}) \otimes E(V^{impair})$$

où $P(V^{pair})$ désigne l'algèbre polynômiale sur les éléments de la base de degré pair et $E(V^{impair})$ l'algèbre extérieure sur les éléments de la base de degré impair. Soit V, W deux espaces vectoriels gradués, alors

$$\Lambda(V \oplus W) = \Lambda V \otimes \Lambda W.$$

Je présente maintenant la définition de modèles de Sullivan, en me limitant aux modèles minimaux ne comportant aucun élément de degré un.

Definition 2.2 Un *modèle de Sullivan* est une ADGC de la forme $(\Lambda V, d)$ avec $V = V^{\geq 2}$, i. e. concentré en degré supérieur ou égal à deux, et telle que

$$d : V^n \to \Lambda(V^{\leq n-1}), n \in \mathbb{N}.$$

Un modèle de Sullivan associé à un espace topologique X est un modèle de Sullivan $(\Lambda V, d)$ tel que

$$(\Lambda V, d) \xrightarrow{\simeq} A(X).$$

Tout espace topologique simplement connexe admet un modèle de Sullivan, unique à isomorphisme près. Exemples :

$$(\Lambda(x_2, z_3), d) \xrightarrow{\simeq} A(S^2) \quad \text{avec} \quad dz_3 = x_2^2.$$

Par $\Lambda(x_2, z_3)$, on désigne l'algèbre commutative libre sur l'espace vectoriel gradué de base, un élément de degré 2, x_2, et un élément de degré 3, z_3.

$$(\Lambda(x_2, y_5), d) \xrightarrow{\simeq} A(\mathbb{CP}^2) \quad \text{avec} \quad dy_5 = x_2^3.$$

Theorem 2.3 *(Sullivan) Deux espaces rationnels simplement connexes de cohomologie de type fini ont le même type d'homotopie si et seulement si ils admettent des modèles de Sullivan isomorphes.*

Soit $F \overset{i}{\hookrightarrow} E \overset{p}{\twoheadrightarrow} B$ une fibration. Le théorème du modèle de la fibre permet de calculer un modèle de Sullivan de F. Soit $(\Lambda Y, d)$ (resp. $(\Lambda X, d)$) un modèle de Sullivan de B (resp. de E). Alors il existe un morphisme d'ADGC

$\Psi : (\Lambda Y, d) \to (\Lambda X, d)$ tel que le carré suivant commute en homologie et donc qui donne $H^*(p)$ en homologie.

$$\begin{array}{ccc} A(B) & \xrightarrow{A(p)} & A(E) \\ \uparrow \simeq & & \uparrow \simeq \\ (\Lambda Y, d) & \overset{\Psi}{\dashrightarrow} & (\Lambda X, d) \end{array}$$

Soit φ le composée $Y \overset{i}{\hookrightarrow} \Lambda Y \overset{\Psi}{\to} \Lambda X \overset{i}{\twoheadrightarrow} X$ où i désigne l'inclusion canonique et p la projection canonique. Alors nous construisons par récurrence sur le degré, une factorisation de Ψ,

$$(\Lambda Y, d) \hookrightarrow (\Lambda Y \otimes \Lambda \mathrm{coker}\varphi \otimes \Lambda s\mathrm{ker}\varphi, D) \underset{p}{\overset{\simeq}{\to}} (\Lambda X, d)$$

où $s\mathrm{ker}\varphi$ est l'espace vectoriel gradué obtenu à partir de $\mathrm{ker}\varphi$ en abaissant les degrés de un. Cette factorisation est en quelque sorte une résolution libre du $(\Lambda Y, d)$-module $(\Lambda X, d)$. Soit $(\Lambda(\mathrm{coker}\varphi \oplus s\mathrm{ker}\varphi), \overline{D})$ le modèle de Sullivan obtenu en divisant l'ADGC $(\Lambda Y \otimes \Lambda \mathrm{coker}\varphi \otimes \Lambda s\mathrm{ker}\varphi, D)$ par l'idéal engendré par Y.

Theorem 2.4 *(Grivel-Thomas-Halperin) Ce modèle de Sullivan est un modèle de Sullivan de F.*

Reprenons l'inclusion $f : S^2 \hookrightarrow \mathbb{CP}^2$. Nous allons calculer un modèle de Sullivan de sa fibre homotopique F. Nous obtenons le diagramme d'ADGC

$$\begin{array}{ccccc} A(\mathbb{CP}^2) & \xrightarrow{A(f)} & A(S^2) & \longrightarrow & A(F) \\ \uparrow \simeq & & \uparrow \simeq & & \uparrow \simeq \\ (\Lambda(x_2, y_5), dy_5 = x_2^3) & \overset{\Psi}{\dashrightarrow} & (\Lambda(x_2, z_3), dz_3 = x_2^2) & & (\Lambda(z_3, sy_5), 0) \\ & \searrow & \uparrow \simeq & \nearrow & \\ & & (\Lambda(x_2, y_5, z_3, sy_5), Dz_3 = x_2^2, Dsy_5 = y_5 - z_3 x_2^3) & & \end{array}$$

Cellulairement, on voit facilement que $H^2(f)$ est un isomorphisme, d'où Ψ envoie bien x_2 sur x_2. En particulier $H^*(F;\mathbb{Q}) \cong E(z_3) \otimes P(z_4)$ comme algèbre , en posant $z_4 = sy_5$.

3 Le théorème du pseudo-modèle de la fibre

On voudrait étendre les résultats de l'homotopie rationnelle en particulier, le théorème du modèle de la fibre sur un corps $\Bbbk$ de caractéristique p quelconque.

Premier problème : en général, à coefficients dans $\Bbbk$, il n'existe pas deux ADG $A(X)$ et $D(X)$ telles que

$$A(X) \overset{\simeq}{\leftarrow} D(X) \overset{\simeq}{\rightarrow} C^*(X)$$

où $A(X)$ est commutative. Le cas peut néanmoins se produire pour certains espaces topologiques, par exemple ceux qui sont dans le domaine d'Anick. Soit r un entier ≥ 1 fixé.

Definition 3.1 Un espace topologique X est *(strictement) dans le domaine d'Anick* si $H^*(X)$ est concentré en degrés $> r$ et $\leq rp$ (respectivement $< rp$).

Le passage à l'ADGC $A(X)$ est fonctoriel. Soit $p : E \to B$ une fibration de fibre F entre deux espaces topologiques dans le domaine d'Anick, on obtient le même diagramme d'ADG que sur $\mathbb{Q}$.

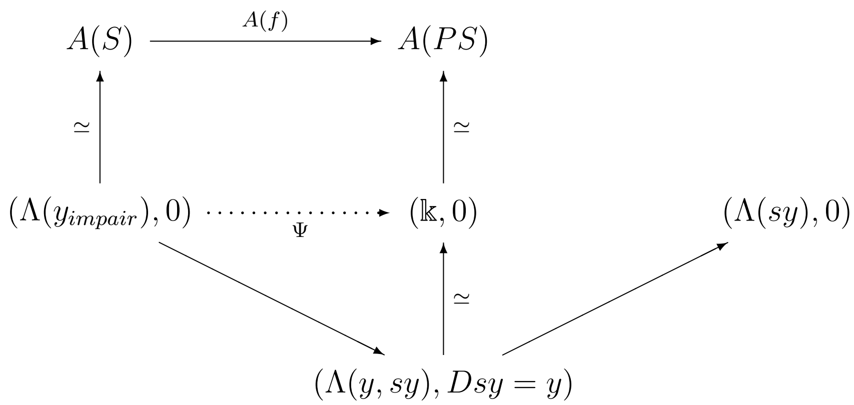
$$\begin{array}{ccc}
C^*(B) & \xrightarrow{C^*(p)} & C^*(E) \\
\uparrow \simeq & & \uparrow \simeq \\
D(B) & \longrightarrow & D(E) \\
\downarrow \simeq & & \downarrow \simeq \\
A(B) & \xrightarrow[A(p)]{} & A(E) \\
\uparrow \simeq & & \uparrow \simeq \\
(\Lambda Y, d) & \overset{\Psi}{\cdots\cdots\rightarrow} & (\Lambda X, d)
\end{array}$$

où $(\Lambda Y, d)$ (resp. $(\Lambda X, d)$) est un modèle de Sullivan de B (resp. de E).

On veut faire une factorisation de Ψ comme en homotopie rationnelle. Essayons dans le cas de la fibration des chemins sur une sphère S de dimension impaire :

$$\Omega S \hookrightarrow PS \overset{p}{\twoheadrightarrow} S.$$

On obtient alors le diagramme commutatif d'ADGC



La factorisation n'est pas valable sur un corps $\mathbb{k}$ de caractéristique p, car

$$d(sy)^p = p(sy)^{p-1}y = 0.$$

Et donc nous n'avons pas $(\Lambda(y, sy), Dsy = y) \xrightarrow{\simeq} (\mathbb{k}, 0)$.

Heureusement, d'ailleurs, car l'algèbre $H^*(\Omega S)$ n'est pas isomorphe à $\Lambda(sy)$. En effet $H^*(\Omega S)$ est l'algèbre à puissances divisées engendrée par un générateur sy de degré pair, notée $\Gamma(sy)$. L'algèbre $\Gamma(sy_{impair})$ est l'algèbre engendrée par les $\gamma^i(sy)$, $i \in \mathbb{N}^*$, avec les relations

$$\gamma^i(v)\gamma^j(v) = \frac{(i+j)!}{i!j!}\gamma^{i+j}(v).$$

En particulier, nous avons $(sy)^p = p!\gamma^p(sy) = 0$. L'algèbre $\Gamma(sy_{pair})$ est, quant à elle, simplement l'algèbre extérieure sur sy. C'est donc $\Lambda(sy)$, car sy est de degré impair. Nous pouvons donc définir l'algèbre à puissances divisées sur l'espace vectoriel gradué V, notée ΓsV, grâce à sa base par la formule

$$\Gamma s(V \oplus W) = \Gamma sV \otimes \Gamma sW$$

où V et W désignent deux espaces vectoriels gradués. Le mieux est de se souvenir que, sur $\mathbb{Q}$, $\Gamma V \cong \Lambda V$ par $\gamma^i(v) \mapsto \dfrac{v^i}{i!}$.

Grâce aux algèbres à puissances divisées, nous pouvons maintenant généraliser sur un corps quelconque, la factorisation de Ψ obtenue sur $\mathbb{Q}$. En effet, Ψ se factorise en

$$(\Lambda Y, d) \hookrightarrow (\Lambda Y \otimes \Lambda \mathrm{coker}\varphi \otimes \Gamma s\mathrm{ker}\varphi, D) \underset{p}{\overset{\simeq}{\twoheadrightarrow}} (\Lambda X, d)$$

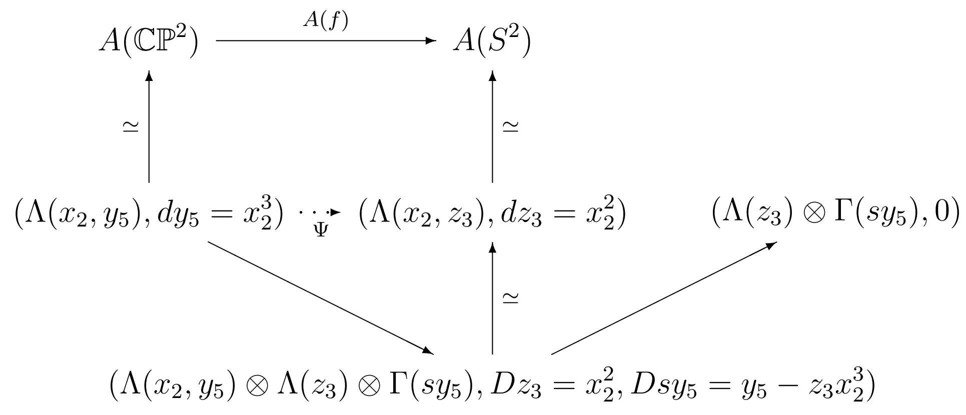
où φ désigne à nouveau le composé $Y \hookrightarrow \Lambda Y \xrightarrow{\Psi} \Lambda X \twoheadrightarrow X$. Soit $(\Lambda(\mathrm{coker}\varphi \oplus s\mathrm{ker}\varphi), \overline{D})$ l'ADGC obtenue en divisant l'ADGC $(\Lambda Y \otimes \Lambda \mathrm{coker}\varphi \otimes \Gamma s\mathrm{ker}\varphi, D)$ par l'idéal engendré par Y.

Theorem 3.2 *Si de plus E est strictement dans le domaine d'Anick, alors cette ADGC a pour cohomologie $H^*(F)$.*

La principale différence avec le théorème du modèle de la fibre rationnel est que généralement, l'ADGC $(\Lambda(\mathrm{coker}\varphi \oplus s\mathrm{ker}\varphi), \overline{D})$ n'est pas reliée à $C^*(F)$ par des morphismes d'ADG induisant des isomorphismes en homologie. De toute façon, E et B peuvent être dans le domaine d'Anick sans qu'il existe une ADGC A et une ADG D telles que $C^*(F) \xleftarrow{\simeq} D \xrightarrow{\simeq} A$. En particulier, F n'est pas forcément dans le domaine d'Anick.

Il y a des contrexemples au théorème, si on ne prend pas à la fois E strictement dans le domaine d'Anick, et B dans le domaine d'Anick.

Nous allons reprendre l'exemple auquel nous avons déjà appliqué le théorème du modèle de la fibre. Mais, cette fois, nous allons utiliser notre théorème du pseudo-modèle de la fibre. Soit $f : S^2 \hookrightarrow \mathbb{CP}^2$ l'inclusion. Notons F sa fibre homotopique. Nous avons le diagramme d'ADGC



Si $p \geq 4$, on obtient alors que $H^*(F; \mathbb{Z}_p) \cong E(z_3) \otimes \Gamma(z_4)$ comme algèbre.

Part II
On the cohomology algebra of a fiber

Abstract

Let $f : E \to B$ be a continuous map between path connected pointed spaces. Let $m(f) : (TX, \partial) \to (TY, \partial')$ be a chain model for $C_*(\Omega f)$ with coefficients in $\mathbb{Z}_p$. Following Anick, (TX, ∂) and (TY, ∂') are (non-associative in general) Hopf algebras and $m(f)$ is, up to homotopy, a Hopf algebra morphism. The main aim of this article is to show how to compute from the above data the cohomology algebra $H^*(F_f; \mathbb{Z}_p)$ of the homotopy fiber F_f of f. A first illustration of the utility of this construction concerns the fiber of a suspended map. A second illustration is a generalization to the "Anick range" of a well-known rational homotopy theorem, asserting that "the fiber of a model is a model of the fiber".

4 Introduction

Let $f : E \to B$ be a continuous map between path connected pointed spaces, F_f its homotopy fiber and $\Bbbk$ a field. In this article, we are interested in the question: how to compute the cohomology algebra $H^*(F_f; \Bbbk)$? More precisely, we would like to exhibit from f an explicit cochain algebra, as "small" as possible (e.g. of finite type when E and B are of finite type) whose cohomology algebra is isomorphic to $H^*(F_f; \Bbbk)$.

When $\Bbbk = \mathbb{Q}$, the field of rational numbers, the Grivel-Thomas-Halperin theorem ([Gri], [Hal1], [FHT2] Th. 14.13) in rational homotopy theory gives the answer.

When $\Bbbk$ is of positive characteristic, the problem remains unsolved, although some partial results have been established. The vector space $H^*(F_f; \Bbbk)$ can be computed via the Eilenberg-Moore spectral sequence, but the differentials are often very intricate. It is also possible to apply the Serre spectral sequence to the homotopy fibration $\Omega B \to F_f \to E$ but again, only the vector space structure of $H^*(F_f; \Bbbk)$ can be captured in general.

Three main attempts have been made to compute $H^*(F_f; \Bbbk)$ as an algebra. The first one by Dupont and Hess [DH]. They have proved that a cochain model for F_f can be obtained from certain cochain models of ΩB and E, but the construction is not explicit. The second one by N'dombol [NDo]. Its construction is rather explicit but it is not known whether it gives the correct algebra structure. The last one by Félix, Halperin and Thomas [FHT1]. They constructed a coproduct on the bar construction $B(C_*(\Omega B;\mathbb{k});C_*(\Omega E;\mathbb{k}))$ and showed that its dual is weakly equivalent to the cochain algebra $C^*(F_f;\mathbb{k})$. Nevertheless this construction, although explicit and with the right cohomology algebra, is far from being small.

Our main idea is to try to replace in the above bar construction the chain Hopf algebras $C_*(\Omega B;\mathbb{k})$ and $C_*(\Omega E;\mathbb{k})$ by their chain models (TX,∂) and (TY,∂') over $\mathbb{k}$, therefore obtaining, after dualization, a smaller cochain algebra $\#B(TY;TX)$ of cohomology the algebra $H^*(F;\mathbb{k})$. A classical example of a chain model (TX,∂) is, in the simply connected case, the Adams-Hilton model of B [AH].

The primary difficulty in constructing the algebra $\#B(TY;TX)$ is that the existence of the coproduct on $B(C_*(\Omega B;\mathbb{k});C_*(\Omega E;\mathbb{k}))$ is based on the fact that $C_*(\Omega f;\mathbb{k})$ is a strict Hopf algebra morphism, which is not true at the model level, i.e. for $m(f)$. Nevertheless, we show (Corollary 9.5) that $B(TY;TX)$ also has a diagonal and that there exists a quasi-isomorphism commuting up to homotopy with the diagonals:

$$B(TY;TX) \xrightarrow{\simeq} B(C_*(\Omega B;\mathbb{k});C_*(\Omega E;\mathbb{k})).$$

This is enough to deduce that $\#B(TY;TX)$ has the right cohomology algebra structure, i.e. is an explicit, small, cochain model for F_f.

Our illustrations concern two particular cases where the chain models (TX,∂) and (TY,∂') are known as Hopf algebras up to homotopy.

The first case is when $f=\Sigma g$ is a suspended map. We show (Theorem 10.2) how to compute the cohomology algebra $H^*(F_{\Sigma g};\mathbb{k})$ in general. In the case where $H_*(g)$ is injective, we give a simple and explicit formula of $H^*(F_{\Sigma g};\mathbb{k})$ (Corollary 10.4).

The second case is when E and B are in the "Anick range"[Ani]. This means that E and B are both r-connected CW-complexes of dimension $\leq rp$. In this case, (TX,∂) and (TY,∂') are isomorphic to enveloping algebras of free differential Lie algebras, equipped with their standard Hopf algebra structure. If moreover $H_{rp}(E)=0$, we deduce (Theorem 12.2) that f admits a "commutative model" and that, as in the rational case, a "fiber" of this commutative model is, up to homotopy, a commutative cochain model of F_f.

Acknowledgments: I wish to thank my supervisor Nicolas Dupont. He spent a lot of time and effort, motivating me on the problem of computing the cohomology algebra of a fiber. I wish also to thank Steve Halperin. He hosted me twice in Toronto, and in particular, gave me Theorem 12.2 to prove, with the counterexample 12.11.

5 Algebraic preliminaries and notations

We work over an arbitrary field $\mathbb{k}$. References for these algebraic preliminaries are [FHT1], [Hal2], [FHT2] and [Ani]. We just give our notations and recall the less-known definitions.

The symbol $\cong$ denotes an isomorphism. The homology functor from differential graded objects to graded objects is denoted H. The denomination "chain" will be restricted to objects with a positive lower degree and "cochain" to those with a positive upper degree. The degree of a element x is denoted $|x|$.

The *suspension* of a graded vector space M is a graded vector space sM such that $(sM)_{i+1} = M_i$.

Let C be an augmented complex. The kernel of the augmentation is denoted $\overline{C}$.

A *differential graded algebra*, or DGA, is a complex A equipped with two morphisms of complexes $\mu : A \otimes A \to A$ and $\eta : \mathbb{k} \to A$ called the *multiplication* and the *unit* such that $\mu \circ (\mu \otimes 1) = \mu \circ (1 \otimes \mu)$ (associativity) and $\mu \circ (\eta \otimes 1) = \mu \circ (1 \otimes \eta)$ (unitary). The commutator isomorphism $\tau : A \otimes B \overset{\cong}{\to} B \otimes A$ is given by $\tau(a \otimes b) = (-1)^{|a||b|} b \otimes a$. A *commutative* DGA or CDGA is a DGA such that $\mu \circ \tau = \mu$.

A *derivation* D in an algebra A is a linear map of degree $|D|$ such that $Dxy = Dx.y + (-1)^{|D||x|} x.Dy$, $x, y \in A$.

The tensor algebra TV on a complex V is the free DGA on V. The free CDGA on V is denoted ΛV.

The *free divided power algebra* ΓV on V is the free CDGA generated by $\gamma^i(v)$ for $v \in V_{even}, i \in \mathbb{N}^*$ and v for $v \in V_{odd}$ divided by the relations $\gamma^i(v)\gamma^j(v) = \dfrac{(i+j)!}{i!j!}\gamma^{i+j}(v)$. Over $\mathbb{Q}$, $\Gamma V \cong \Lambda V$ by $\gamma^i(v) \mapsto \dfrac{v^i}{i!}$.

Let A be a DGA, M a right A-module, N a left A-module. The tensor product of M and N over A, denoted $M \otimes_A N$, is the complex quotient of $M \otimes N$ by the sub-complex generated by $m.a \otimes n - m \otimes a.n$, $m \in M$, $n \in N$,

$a \in A$.

Definition 5.1 An *(augmented) A-algebra* B is a morphism of (augmented) DGA's $f : A \to B$ such that $f(A)$ is the center of B. Let $f : A \to B$, $g : A \to C$ be two (augmented) A-algebras. A morphism of (augmented) DGA's $\varphi : B \to C$ is a *morphism of (augmented) A-algebras* if $g = \varphi \circ f$.

If A is an (augmented) CDGA, A is an (augmented) A-algebra by considering $id : A \to A$.

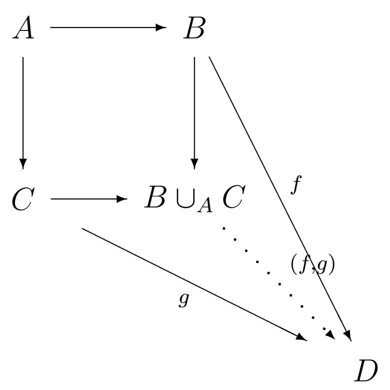
Property 5.2 Let $f : A \to B$, $g : A \to C$ be two DGA morphisms.

(i) If C is an A-algebra, then $B \otimes_A C$ is the quotient of the DGA $B \otimes C$ by the left ideal generated by $f(a) \otimes 1 - 1 \otimes g(a)$.

(ii) If A is augmented, $B \otimes_A \mathbb{k}$ is the quotient of B by the left ideal generated by $f(\overline{A})$.

(iii) If B and C are both A-algebras, $B \otimes_A C$ is an A-algebra.

Let $A \to B$, $A \to C$ be two morphisms in a category. If it exists, the push out and the morphism given by the universal property will be denoted as in the commutative diagram:



If it exists, the sum of B and C is denoted $B \coprod C$.

The push out exists in the category of DGA's. The push out is $B \otimes_A C$ in the category of CDGA's. In particular, the tensor product of DGA's is the sum in the category of CDGA's.

A *differential graded coalgebra*, or DGC, is a complex C equipped with two morphisms of complexes $\Delta : C \to C \otimes C$ and $\varepsilon : C \to \Bbbk$ called the *diagonal* and the *counit* such that $(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$ (coassociativity) and $(\varepsilon \otimes 1) \circ \Delta = 1 = (1 \otimes \varepsilon) \circ \Delta$ (counitary). A DGC is *cocommutative* if $\tau \circ \Delta = \Delta$. The dual $Hom(C, \Bbbk)$ of a DGC C is a DGA denoted $\#C$.

A *differential graded Hopf algebra*, or DGH, is a DGA K equipped with two DGA morphisms $\Delta : K \to K \otimes K$ and $\varepsilon : K \to \Bbbk$ such that $(\Delta \otimes 1) \circ \Delta = (1 \otimes \Delta) \circ \Delta$ and $(\varepsilon \otimes 1) \circ \Delta = 1 = (1 \otimes \varepsilon) \circ \Delta$.

The notion of homotopy we use in the category of augmented DGA's, is recalled in section 7. The symbol $\approx$ stands for homotopic morphisms.

A *Hopf algebra up to homotopy*, or HAH, is an augmented DGA K equipped with a morphism of augmented DGA's $\Delta : K \to K \otimes K$ such that $(\Delta \otimes 1) \circ \Delta \approx (1 \otimes \Delta) \circ \Delta$ and $(\varepsilon \otimes 1) \circ \Delta \approx 1 \approx (1 \otimes \varepsilon) \circ \Delta$. Let K, K' be two HAH's. A morphism of augmented DGA's $f : K \to K'$ is a *HAH morphism* if $\Delta f \approx (f \otimes f)\Delta$.

Let K be a DGH. A left K*-coalgebra* D is both a DGC and a left K-module such that the action $K \otimes D \to D$ is a DGC morphism.

Property 5.3 The tensor product of a right K-coalgebra C and a left K-coalgebra D over K, $C \otimes_K D$, is a DGC.

A *differential graded Lie algebra*, or DGL, is a complex L equipped with a morphism of complexes: $[\ ,\ ] : L \otimes L \to L$ such that for x, y, $z \in L$:

- $[x, y] = -(-1)^{|x||y|}[y, x]$
- $(-1)^{|x||z|}[x, [y, z]] + (-1)^{|z||y|}[z, [x, y]] + (-1)^{|y||x|}[y, [z, x]] = 0$
- $[x, [x, x]] = 0$, $x \in L_{odd}$

The universal enveloping algebra of L is denoted UL.

A quasi-isomorphism is denoted $\xrightarrow{\simeq}$. Two objects A and B in a category $\mathcal{C}$ are *(naturally) weakly* $\mathcal{C}$*-equivalent*, denoted $A \sim B$, if they are connected by a chain of (natural) $\mathcal{C}$-quasi-isomorphisms of the form:

$$A \xleftarrow{\simeq} A(1) \xrightarrow{\simeq} \cdots\cdots \xleftarrow{\simeq} A(n) \xrightarrow{\simeq} B.$$

Let A be an augmented DGA, M a left A-module, N a right A-module. Denote d_1 be the differential of the complex $M \otimes T(s\overline{A}) \otimes N$ obtain by tensorization. We denote the tensor product of the elements $m \in M$, $sa_1 \in$

$s\overline{A}$, ..., $sa_1 \in s\overline{A}$, $n \in N$ by $m[sa_1|\cdots|sa_k]n$. Let d_2 be the differential on the graded vector space $M \otimes T(s\overline{A}) \otimes N$ defined by:

$$\begin{aligned} d_2 m[sa_1|\cdots|sa_k]n &= (-1)^{|m|} m a_1[sa_2|\cdots|sa_k]n \\ &+ \sum_{i=1}^{k-1} (-1)^{\varepsilon_i} m[sa_1|\cdots|sa_i a_{i+1}|\cdots|sa_k]n \\ &- (-1)^{\varepsilon_{k-1}} m[sa_1|\cdots|sa_{k-1}]a_k n; \end{aligned}$$

Here $\varepsilon_i = |m| + |sa_1| + \cdots + |sa_i|$.

Remark 5.4 We only find the above formula in the non graded case in the literature ([McL] X.(2.5)). We obtain the appropriate signs by Mac Lane's condensation of complexes of complexes ([McL] X.9). If we set $N = \Bbbk$, we rediscover the same formula as in [FHT1] §4.

The *bar construction of* A *with coefficients in* M *and* N, denoted $B(M; A; N)$, is the complex $(M \otimes T(s\overline{A}) \otimes N, d_1 + d_2)$. We use mainly $B(M; A) = B(M; A; \Bbbk)$. The *reduced bar construction of* A, denoted $B(A)$, is $B(\Bbbk; A)$.

Let L be a DGL, M a left UL-module. If $\frac{1}{2} \in \Bbbk$, $B(M; UL)$ has a subcomplex $C_*(M; L) = (M \otimes \Gamma sL, d_1 + d_2)$. Its dual, denoted $C^*(M; L)$, is called the *Cartan-Chevalley-Eilenberg complex with coefficients.* Again $C_*(L)$ denotes $C_*(\Bbbk; L)$.

Let A be a DGA. A *semifree extension of an* A*-module* M is an inclusion of A-modules: $(M, d) \rightarrowtail (M \oplus (A \otimes V), D)$ such that:

- $V = \oplus_{k \in \mathbb{N}} V(k)$ as graded vector space.
- $D : V(k) \rightarrow M \oplus (A \otimes V(< k)), k \in \mathbb{N}$ where $V(< k) = \oplus_{i=0}^{k-1} V(i)$

A A*-semifree module* is an A-module $(A \otimes V, D)$ such that $0 \rightarrowtail (A \otimes V, D)$ is a semifree extension of 0.

A *(augmented) free extension* is an inclusion of (augmented) DGA's: $(A, d) \rightarrowtail (A \coprod TV, D)$ such that $V = \oplus_{k \in \mathbb{N}} V(k)$ and $D : V(k) \rightarrow A \coprod TV(< k), k \in \mathbb{N}$

A *free DGA* is a DGA (TV, D) such that $\Bbbk \rightarrowtail (TV, D)$ is a free extension.

A *(augmented) relative Sullivan model* is an inclusion of (augmented) CDGA's: $(A, d) \rightarrowtail (A \otimes \Lambda V, D)$ such that: $V = \oplus_{k \in \mathbb{N}} V(k)$ and $D : V(k) \rightarrow A \otimes \Lambda V(< k), k \in \mathbb{N}$

A *Sullivan model* is a CDGA $(\Lambda V, D)$ such that $\Bbbk \rightarrowtail (\Lambda V, D)$ is a relative Sullivan model.

Property 5.5 (i) If $A \rightarrowtail A \coprod TV$ is a free extension then $A \coprod TV$ is (left and right) A-semifree.

(ii) If $A \rightarrowtail A \otimes \Lambda V$ is a relative Sullivan model then $A \otimes \Lambda V$ is A-semifree.

The normalized singular chain complex of a topological space X with coefficients in $\Bbbk$ is denoted $C_*(X)$.

Let G be a topological monoid. A *G-fibration* is a surjective fibration $p : E \twoheadrightarrow B$ such that E is a right G-space, for $b \in B$, the *fiber* $p^{-1}(b)$ is stable by G and for $z \in E$, the map $g \mapsto z.g$ is a weak equivalence from G to $p^{-1}(p(z))$.

6 The bar construction with coefficients as a DGC

In this section, we give a simple form of the Félix-Halperin-Thomas diagonal on the bar construction ([FHT1] 4.1), analogue to the definition of the diagonal on $C_*(X)$ ([McL] p. 245). We then review some of the results of [FHT1] and give an immediate application (Proposition 6.7).

Property 6.1 ([McL] X.7.2) Let A (resp. B) be an augmented DGA, M (resp. N) a right A-module (resp. B-module) and P (resp. Q) a left A-module (resp. B-module). Then we have an Alexander-Whitney morphism of complexes

$$AW : B(P \otimes Q; A \otimes B; M \otimes N) \to B(P; A; M) \otimes B(Q; B; N)$$

where the image of a typical element $p \otimes q[s(a_1 \otimes b_1)|\cdots|s(a_k \otimes b_k)]m \otimes n$ is

$$\sum_{i=0}^{k} (-1)^{\zeta_i} p[sa_1|\cdots|sa_i]a_{i+1}\cdots a_k m \otimes qb_1 \cdots b_i[sb_{i+1}|\cdots|sb_k]n.$$

$$\text{Here} \quad \zeta_i = \sum_{j=1}^{n} \left(|b| + \sum_{l=1}^{j-1} |b_l|\right)|a_j| + \left(|b| + \sum_{l=1}^{j-1} |b_l|\right)|m| + \sum_{j=i+1}^{n} (j-i)|a_j| + (n-i)|m| + |i||b| + \sum_{j=1}^{i-1} (i-j)|b_j|.$$

AW is natural, associative ($AW \circ (AW \otimes id) = AW \otimes (id \otimes AW)$).

Remark 5.4 holds here too.

Corollary 6.2 *Let K be a DGH, C a right K-coalgebra, D a left K-coalgebra. Then $B(C;K;D)$ is a DGC with the diagonal*

$$B(C;K;D) \xrightarrow{B(\Delta_C;\Delta_K;\Delta_D)} B(C\otimes C;K\otimes K;D\otimes D) \xrightarrow{AW} B(C;K;D)\otimes B(C;K;D)$$

$$\textit{and the counit} \quad B(C;K;D) \xrightarrow{B(\varepsilon_C;\varepsilon_K;\varepsilon_D)} B(\mathbb{k};\mathbb{k};\mathbb{k}) = \mathbb{k}.$$

This coalgebra structure on $B(C;K;D)$ is functorial.

Proof. It is obvious with commutative diagrams using AW's associativity, naturality and the functoriality of the bar construction. *QED*

Property 6.3 If C is a right K-semifree-coalgebra then $B(C;K;D) \xrightarrow{\simeq} C\otimes_K D$ is a DGC quasi-isomorphism.

Remark 6.4 For $B(C;K)$, the coalgebra structure coincides with the one defined in ([FHT1] 4.1). The proof is a tedious calculation. Anyway, we don't need to give it, since we will verify that the following theorem is valid independently of the functorial coalgebra structure chosen on the bar construction, either the one defined by Félix-Halperin-Thomas, or the one defined in Corollary 6.2.

Theorem 6.5 *([FHT1] 5.1) Let $p: E \to B$ be a G-fibration. Then there is a natural DGC quasi-isomorphism $B(C_*(E);C_*(G)) \xrightarrow{\simeq} C_*(B)$.*

Proof. • As shown in Theorem 8.3 [FHT2], if $m: M \xrightarrow{\simeq} C_*(E)$ is a right $C_*(G)$-semifree resolution of $C_*(E)$, then we have the commuting diagram of complexes.

$$\begin{array}{ccc} M & \xrightarrow[\simeq]{m} & C_*(E) \\ \zeta \downarrow & & \downarrow C_*(p) \\ M\otimes_{C_*(G)}\mathbb{k} & \xrightarrow[\overline{m}]{\simeq} & C_*(B) \end{array}$$

• We can take $M = B(C_*(E);C_*(G);C_*(G))$. Since m, $C_*(p)$ and ζ are DGC morphisms and ζ is an epimorphism, $\overline{m}$ is a DGC morphism too. *QED*

Actually, we can generalize Theorem 6.5 to $B(C_*(E);C_*(G);C_*(Y))$ where Y is another G-space.

Proposition 6.6 *([FHT1] 6.7) Let $f : E \to B$ be a continuous map with B path connected, F_f its homotopy fiber then there is a natural DGC quasi-isomorphism $B(C_*(F_f); C_*(\Omega B)) \xrightarrow{\simeq} C_*(E)$.*

Proof. Since B is path connected, the Moore path space fibration $PB \twoheadrightarrow B$ is an ΩB-fibration. So, by pull back, we obtain an ΩB-fibration $p_0 : F_f \to E$. We apply the Theorem 6.5 to p_0. *QED*

Proposition 6.7 *Let $f : E \to B$ be a continuous pointed map with E path connected. Then $C_*(F_f)$ is naturally weakly DGC equivalent to $B(C_*(\Omega B); C_*(\Omega E))$.*

Proof. We have the morphism of topological monoids $\Omega f : \Omega E \to \Omega B$. So ΩB is an ΩE-space and $C_*(\Omega f) : C_*(\Omega E) \to C_*(\Omega B)$ is a DGH morphism. There is a natural morphism of ΩE-spaces $F_{p_0} \xrightarrow{\simeq} \Omega B$ which is a homotopy equivalence and apply now Proposition 6.6 to p_0. *QED*

7 Homotopy of chain complexes and of augmented DGA's

We recall the notion of homotopy of chain complexes and of augmented DGA's using cylinders since our proof will rely heavily on.

Property 7.1 (i) The category of chain complexes is a proper model category where cofibrations are $\mathbb{k}$-semifree extensions and fibrations surjections.

(ii) The category of augmented DGA's is a proper model category where cofibrations are free extensions ([FHT1] §3) and fibrations surjections.

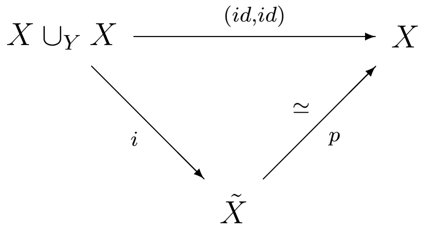
The notion of homotopy can be defined in any proper model category ([Qui]), but is quite complicated in general. In the category of chain complexes, it is simple since for every complex X, $0 \rightarrowtail X$ is a cofibration (X is *cofibrant*). In the category of augmented DGA's, Felix, Halperin and Thomas have found a helpful simplification.

In this section, $Y \to X$ is going to be either

- a cofibration and we define relative homotopy and follow [Bau1] in the case of any proper model category where surjections are fibrations.

- or just the unit of X, $\Bbbk \to X$ and we define absolute homotopy for augmented DGA's, and follow [FHT1] §3.

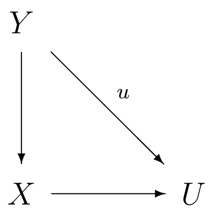
Definition 7.2 ([Maj] 1.1) An object $\tilde{X}$ is a *left homotopy object* if there is a factorization of the folding map



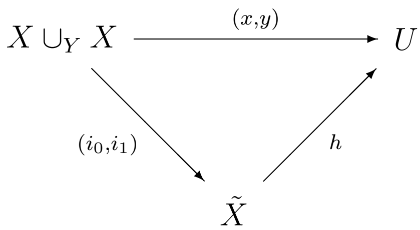
Remark 7.3 Let i_0 (resp. i_1) the composite of the first (resp. second) inclusion $X \to X \cup_Y X$ with i. Then by universal property, $i = (i_0, i_1)$ and we use this last notation.

Definition 7.4 A *cylinder* $I_Y X$ is a left homotopy object such that (i_0, i_1) is a cofibration.

Let $u : Y \to U$ be a fixed morphism. Let $x, y : X \to U$ be two morphisms such that for both of them the following diagram commutes:



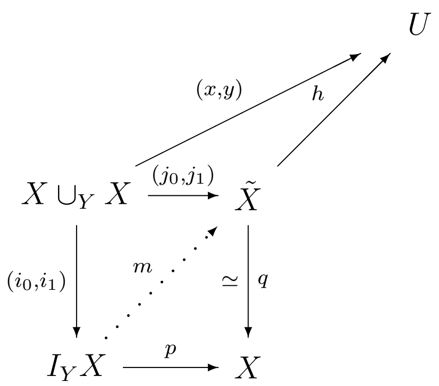
Definition 7.5 x and y are *homotopic* for the left homotopy object $\tilde{X}$ if there is a commutative diagram



We call h a *homotopy* from x to y, and denoted it $h : x \approx y$.

Property 7.6 If we fix a cylinder $I_Y X$, then for any homotopy $h : x \approx y$ starting from a left homotopy object $\tilde{X}$, there exists a homotopy $h \circ m : x \approx y$ starting from $I_Y X$. In particular, all cylinders define the same notion of homotopy between morphisms.

Proof. Just apply the lifting lemma to the diagram



QED

Property 7.7 The homotopy relation defined with a cylinder is an equivalence relation. The equivalence classes are called homotopy classes relative to u or Y. The homotopy class of x is denoted $[x]$. The set of classes is denoted $[X, U]^u$ or $[X, U]^Y$.

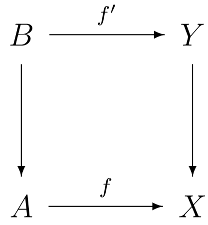
Definition 7.8 (i) The homotopy $x \circ p : x \approx x$ is called the *trivial homotopy* and is denoted 0.

(ii) Let $h : x \approx y$ and $g : y \approx z$ be two homotopies for the same cylinder. The push out of two cylinders is a left homotopy object and now by Property 7.6, we have a homotopy $(h, g) \circ m : x \approx z$, called the *sum of the homotopies* and denoted $h + g$.

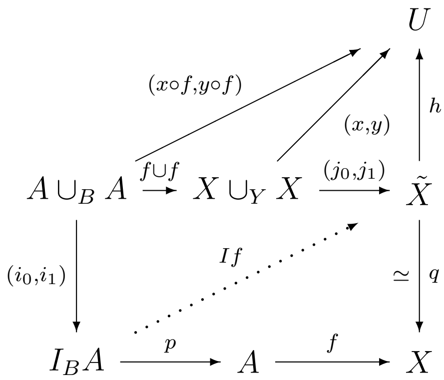
Property 7.9 The notion of homotopy is stable by composition.

Proof. • Let $g : U \to V$ be a morphism. Then $g \circ h : g \circ x \approx g \circ y$ is a homotopy.

• Let



be any commutative diagram with $B \rightarrowtail A$ a cofibration or the unit of A. Then by the lifting lemma, we obtain a morphism $If : I_B A \to \tilde{X}$ such that the following diagram commutes



So $h \circ If : x \circ f \approx y \circ f$ is the desired homotopy. QED

Definition 7.10 ([Bau1] II§5) We choose a cylinder IX. A homotopy $h : x \approx y$ is a morphism $IX \to U$ such that $h \circ (i_0, i_1) = (x, y)$. Since (i_0, i_1) is a cofibration, we can consider the set of homotopy classes relative to (x, y) of such homotopies, denoted $[IX, U]^{(x,y)}$. The elements $[h]$ of the set $[IX, U]^{(x,y)}$ are called *tracks from x to y*.

Definition 7.11 Let X, U be two augmented DGA's. Then $Hom(X, U)$ is a groupoid called the *groupoid of homotopies* with the morphisms from x to y being the tracks from x to y and the composition being $[h] + [g] = [h + g]$ and the identity morphism being $[0]$.

Property 7.12 (i) Let $g : U \to V$ be a morphism. Then the homotopy class $[gh]$ depends only on g and $[h]$.

(ii) Let $f : A \to X$ be a morphism. Then the homotopy class $[hIf]$ depends only on f and $[h]$.

8 Bar construction and homotopies

The notion of homotopy for augmented DGA's can also be defined with (f, g)-derivations ([FHT1] 3.5). But using Félix-Halperin-Thomas diagonal on the bar construction as recalled in Corollary 6.2, and the notion of homotopy defined with cylinders, we now obtain a quick proof of our main result. Indeed, in this section, the proofs are just diagrams and use only the fact that the bar construction is a functor preserving quasi-isomorphisms from the category of pairs of augmented DGA's to the category of chain complexes. On the contrary, the proofs, even for the following property, are long and tedious computations if we use the explicit formula of the diagonal of the bar construction and (f, g)-derivations.

Except where specified, all the morphisms and diagrams are in the category of chain augmented DGA's. Let $f : A \to M$ be a morphism. To specify that the A-module structure on M is given by f, we denote (sometimes) the A-module M, M_f in $B(M; A)$.

Property 8.1 Let

$$\begin{array}{ccc} A & \xrightarrow{\theta:\varphi\approx\varphi'} & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ M & \xrightarrow[\theta':\Psi\approx\Psi']{} & M' \end{array}$$

be a "diagram" where θ and θ' are homotopies, and where $\Psi \circ f = g \circ \varphi$, and $\Psi' \circ f = g \circ \varphi'$. Let $If : IA \to IM$ be a morphism as in the diagram of the proof of 7.9. If $\theta' \circ If = g \circ \theta$ (*naturality of the homotopies*) then the morphisms of chain complexes $B(\Psi; \varphi)$ and $B(\Psi'; \varphi')$ are homotopic.

Proof. Since B preserves quasi-isomorphisms ([FHT1] 4.3(iii)), $B(IM_{If}; IA)$ is a left homotopy object in the category of chain complexes. So $B(\theta'; \theta) : B(\Psi; \varphi) \approx B(\Psi'; \varphi')$ is a homotopy. *QED*

Definition 8.2 Let $x, y : X \to U$ be two morphisms of augmented chain DGA's. Let $h : x \approx y$ be a homotopy for the left homotopy object $\tilde{X}$. We denote by $B(h)$ the quasi-isomorphisms of chain complexes from $B(U_x; X)$ to $B(U_y; X)$ defined uniquely up to homotopy by

$$B(U; i_1) \circ B(h) \approx B(U; i_0).$$

Property 8.3 Let $h \circ m$ be a homotopy corresponding to h for the cylinder IX (Property 7.6). Then $B(h)$ and $B(h \circ m)$ are homotopic.

Proof. For $\varepsilon = 0, 1$, $B(U; j_\varepsilon) = B(U; m) \circ B(U; i_\varepsilon)$. $\boxed{QED}$

Property 8.4 Our $B(h)$ defines a functor from the groupoid of homotopies to the homotopical category of chain complexes:

$$\begin{array}{rcl} x : X \to U & \mapsto & B(U_x; X) \\ [h] : x \approx y & \mapsto & [B(h)] : B(U_x; X) \xrightarrow{\simeq} B(U_y; X) \end{array}$$

Proof. It is easy to see that $[B(h)]$ depends only of $[h]$. So $[B(h)]$ is denoted by $B[h]$. Of course $B[0] = id$. It is also easy to prove that $B[(h, g)] \approx B(h) \circ B(g)$. We apply now Property 8.3. $\boxed{QED}$

Now we show that $B(h)$ "behaves well" with compositions.

Lemma 8.5 *(i) Let* $A \xrightarrow{f} M \xrightarrow{h:\Psi \approx \Psi'} M'$. *Then*

$$B(\Psi'; A) : B(M_f; A) \to B(M'_{\Psi' f}; A)$$

is homotopic to the composite

$$B(M_f; A) \xrightarrow{B(\Psi;A)} B(M'_{\Psi f}; A) \xrightarrow{B(hIf)} B(M'_{\Psi' f}; A).$$

(ii) Let $A \xrightarrow{h:\varphi \approx \varphi'} A' \xrightarrow{g} M'$. *Then*

$$B(M'; \varphi) : B(M'_{g\varphi}; A) \to B(M'_g; A')$$

is homotopic to the composite

$$B(M'_{g\varphi}; A) \xrightarrow{B(gh)} B(M'_{g\varphi'}; A) \xrightarrow{B(M';\varphi')} B(M'_g; A).$$

(iii) Let $A \xrightarrow{h:f \approx f'} M \xrightarrow{\Psi} M'$. *Then the two composites*

$$B(M_f; A) \xrightarrow{B(h)} B(M_{f'}; A) \xrightarrow{B(\Psi;A)} B(M'_{\Psi f'}; A) \quad \textit{and}$$

$$B(M_f; A) \xrightarrow{B(\Psi;A)} B(M'_{\Psi f}; A) \xrightarrow{B(\Psi h)} B(M'_{\Psi f'}; A) \quad \textit{are homotopic.}$$

(iv) Let $A \xrightarrow{\varphi} A' \xrightarrow{h:g\approx g'} M'$. *Then the two composites*

$$B(M'_{g\varphi}; A) \xrightarrow{B(M';\varphi)} B(M'_g; A') \xrightarrow{B(h)} B(M'_{g'}; A') \quad \textit{and}$$

$$B(M'_{g\varphi}; A) \xrightarrow{B(hI\varphi)} B(M'_{g'\varphi}; A) \xrightarrow{B(M';\varphi)} B(M'_{g'}; A') \quad \textit{are homotopic.}$$

Proof. (i) We apply Property 8.1 to the diagram

$$\begin{array}{ccc} A & \xrightarrow{id:i_0\approx i_1} & IA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle h\circ If} \\ M & \xrightarrow[h:\Psi\approx\Psi']{} & M' \end{array}$$

The proofs of (ii), (iii) and (iv) are based on three straightforward commuting diagrams. $\boxed{QED}$

Definition 8.6 Let

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ M & \xrightarrow[\Psi]{} & M' \end{array}$$

be a commuting diagram up to a homotopy $h : \Psi \circ f \approx g \circ \varphi$. We denote by $B(\Psi; \varphi; h)$ the morphisms of chain complexes from $B(M_f; A)$ to $B(M'_g; A')$ defined by

$$B(\Psi; \varphi; h) = B(M'; \varphi) \circ B(h) \circ B(\Psi; A).$$

Now we give a proposition to recognize whenever two $B(\Psi; \varphi; h)$'s are homotopic.

Proposition 8.7 *Let*

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ M & \xrightarrow[\Psi]{} & M' \end{array}$$

be a commuting diagram up to a homotopy $h : \Psi \circ f \approx g \circ \varphi$*. Let* $q : \varphi \approx \varphi'$ *and* $q' : \Psi \approx \Psi'$ *two homotopies. Then*

$$\begin{array}{ccc} A & \xrightarrow{\varphi'} & A' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ M & \xrightarrow[\Psi']{} & M' \end{array}$$

is a commuting diagram up to a homotopy $h' = -q'If + h + gq$*, and the two maps* $B(\Psi; \varphi; h)$*,* $B(\Psi'; \varphi'; h')$ *from* $B(M_f; A)$ *to* $B(M'_g; A')$ *are homotopic.*

Proof. By transitivity, $\Psi' f \approx \Psi f \approx g\varphi \approx g\varphi'$. By property 8.4 and Lemma 8.5 (i) and (ii), we have the desired homotopy. $\boxed{QED}$

Now we give a proposition to compose $B(\Psi; \varphi; h)$'s.

Proposition 8.8 *Let*

$$\begin{array}{ccccc} A & \xrightarrow{\varphi} & A' & \xrightarrow{\varphi'} & A'' \\ {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f''} \\ M & \xrightarrow[\Psi]{} & M' & \xrightarrow[\Psi']{} & M'' \end{array}$$

be two homotopical commuting squares with the homotopies $h : \Psi \circ f \approx f' \circ \varphi$ *and* $h' : \Psi' \circ f' \approx f'' \circ \varphi'$*. Then*

$$\begin{array}{ccc} A & \xrightarrow{\varphi' \circ \varphi} & A'' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f''} \\ M & \xrightarrow[\Psi' \circ \Psi]{} & M'' \end{array}$$

is a commuting diagram up to a homotopy $h'' = \Psi' h + h' I \varphi$ *and*

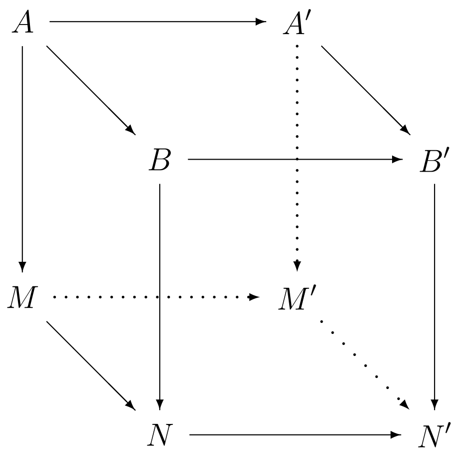
$$B(\Psi' \circ \Psi; \varphi' \circ \varphi; h'') : B(M_f; A) \to B(M''_{f''}; A'')$$

is homotopic to the composite

$$B(M_f;A) \xrightarrow{B(\Psi;\varphi;h)} B(M'_{f'};A') \xrightarrow{B(\Psi';\varphi';h')} B(M''_{f''};A'').$$

Proof. By transitivity, $\Psi'\Psi f \approx \Psi' f'\varphi \approx f''\varphi'\varphi$. By property 8.4 and Lemma 8.5 (iii) and (iv), we have the desired homotopy. *QED*

Our functor transforms a homotopy commutative square into an arrow. We need to know what is the condition such that it changes a homotopy commutative cube into a homotopy commutative square. When we work with a specified morphism from X to Y, we denote it by $[XY]$. Let



be a homotopy commutative cube with a specified homotopy for each face:

$$\begin{aligned}
h_{left} : [MN][AM] &\approx [BN][AB] \\
h_{front} : [NN'][BN] &\approx [B'N'][BB'] \\
h_{up} : [BB'][AB] &\approx [A'B'][AA'] \\
h_{right} : [B'N'][A'B'] &\approx [M'N'][A'M'] \\
h_{back} : [A'M'][AA'] &\approx [MM'][AM] \\
h_{down} : [M'N'][MM'] &\approx [NN'][MN]
\end{aligned}$$

$[NN'][MN][AM]$ is homotopic to itself by two homotopies:

- the trivial homotopy, denoted 0,
- a homotopy involving the homotopy in each face of the cube namely:

$$[NN']h_{left} + h_{front}I[AB] + [B'N']h_{up} + h_{right}I[AA'] + [M'N']h_{back} + h_{down}I[AM].$$

Definition 8.9 These homotopies of the cube are *compatible* if the previous two homotopies are homotopic.

Theorem 8.10 *If the homotopies of the cube are compatible, then we have the homotopy commutative square*

$$\begin{array}{ccc}
B(M;A) & \xrightarrow{B([MM'];[AA'];h_{back})} & B(M';A') \\
\Big\downarrow {\scriptstyle B([MN];[AB];h_{left})} & & \Big\downarrow {\scriptstyle B([M'N'];[A'B'];h_{right})} \\
B(N;B) & \xrightarrow{B([NN'];[BB'];h_{front})} & B(N';B')
\end{array}$$

Proof. We use twice Proposition 8.8 to compose

$$B([NN'];[BB'];h_{front}) \circ B([MN];[AB];h_{left}) \quad \text{and}$$

$$B([M'N'];[A'B'];h_{right}) \circ B([MM'];[AA'];h_{back})$$

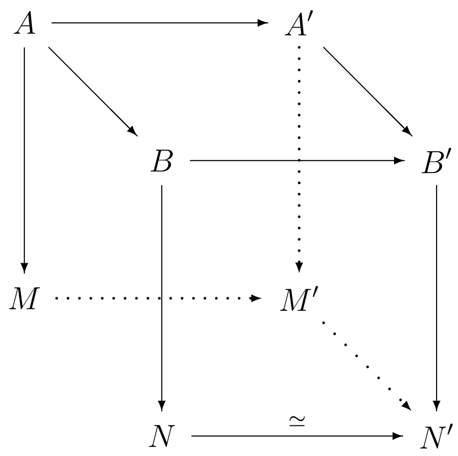
and we apply Proposition 8.7 to see when they are homotopic. *QED*

We give now some properties useful to establish cubes with compatible homotopies.

Property 8.11 If all the faces except the left one commute up to homotopies with given homotopies and $[NN']$ is a quasi-isomorphism, then the left face commutes up to homotopy, with a homotopy h_{left} such that the homotopies are compatible. The homotopy class of h_{left} is determined uniquely by the compatibility condition.

Proof. By the lifting lemma ([Bau1] II.2.11.a), $[NN']$ induces a bijection from the tracks from $[MN][AM]$ to $[BN][AB]$, to the tracks from $[NN'][MN][AM]$ to $[NN'][BN][AB]$. The image of $[h]$ is $[NN'][h]$. $[NN'][h_{left}]$ is now given by the compatibility condition. *QED*

We consider the following homotopy commutative diagram with given homotopies in each square.

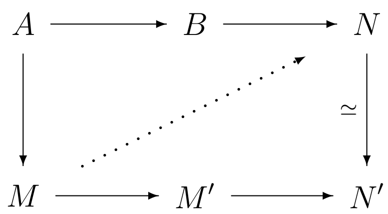


such that

- A and M are free DGA's
- $[AM]$ is a cofibration
- $[NN']$ is quasi-isomorphism.

Property 8.12 The previous diagram can be completed in a homotopy commutative cube with compatible homotopies such that h_{left} is the trivial homotopy (In particular, the left face commutes exactly).

Proof. The diagram



commuts, with homotopy

$$h = -(h_{front}I[AB] + [B'N']h_{up} + h_{right}I[AA'] + [M'N']h_{back}).$$

By Lemma 3.6 [FHT1], there exists $[MN]$ extending $[AN]$ and such that $h_{down}I[AM] = h$. $\boxed{QED}$

Remark 8.13 We could have changed $[MM']$ by a homotopic map such that the back face commutes exactly.

9 Bar construction of Hopf algebras up to homotopy

Property 9.1 Let x, $y : X \to U$ be two morphisms, and $h : x \approx y$ a homotopy. Let x', $y' : X' \to U'$ be two morphisms, and $h' : x' \approx y'$ a homotopy. Then $h \otimes h' : x \otimes x' \approx y \otimes y'$ is a homotopy. The notion of absolute homotopy is stable under tensorization.

Proof. The tensor product of two left homotopy objects is a left homotopy object. *QED*

Remark 9.2 The tensor product of two cylinders is not a cylinder. That is one the reasons why we have introduced left homotopy objects.

Property 9.3 Let

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ M & \xrightarrow[\Psi]{} & N \end{array} \qquad \begin{array}{ccc} A' & \xrightarrow{\varphi'} & B' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle g'} \\ M' & \xrightarrow[\Psi']{} & N' \end{array}$$

be two homotopy commutative diagrams, with the homotopies $h : \Psi f \approx g\varphi$ and $h' : \Psi' f' \approx g'\varphi'$. Then we have the homotopy commutative diagram:

$$\begin{array}{ccc} B(M \otimes M'; A \otimes A') & \xrightarrow{B(\Psi\otimes\Psi';\varphi\otimes\varphi';h\otimes h')} & B(N \otimes N'; B \otimes B') \\ {\scriptstyle AW}\downarrow & & \downarrow{\scriptstyle AW} \\ B(M;A) \otimes B(M';A') & \xrightarrow[B(\Psi;\varphi;h)\otimes B(\Psi';\varphi';h')]{} & B(N;B) \otimes B(N';B') \end{array}$$

Otherwise stated, AW is "natural up to homotopy" with the $B(\Psi;\varphi;h)$'s.

Proof. By definition of the $B(\Psi; \Phi; h)$'s, we have only to prove that

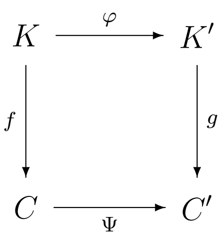
$$(B(h) \otimes B(h')) \circ AW \approx AW \circ B(h \otimes h')$$

and it follows from the definition of the $B(h)$'s. $\boxed{QED}$

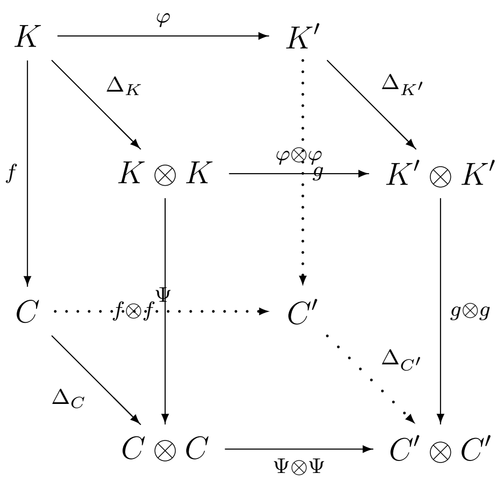
Theorem 9.4 • *Let $f : K \to C$ be a morphism of chain HAH's with given homotopy $h : \Delta_C f \approx (f \otimes f)\Delta_K$. Then $B(C; K)$ has a diagonal:*

$$B(C; K) \xrightarrow{B(\Delta_C;\Delta_K;h)} B(C \otimes C; K \otimes K) \xrightarrow{AW} B(C; K) \otimes B(C; K)$$

• *Let*



be a homotopy commutative diagram of chain HAH's, with a given homotopy $h_{back} : \Psi f \approx g\varphi$ such that the homotopies of the following cube are compatible:



Then $B(\Psi; \varphi; h_{back}) : B(C; K) \to B(C'; K')$ commutes with the diagonals up to homotopy.

Proof. We apply Theorem 8.10 and Property 9.3. $\boxed{QED}$

Let X a graded vector space. I denote a DGA of the form (TX, ∂) simply by TX except when the differential ∂ can be specified. In particular, a free DGA with zero differential is still denoted by $(TX, 0)$.

Corollary 9.5 *Let $f : E \to B$ be a map between two pointed topological spaces with E path connected, F its homotopy fiber. We consider a homotopy commutative diagram of chain algebras as follows:*

$$\begin{array}{ccc} TX & \xrightarrow{\simeq} & C_*(\Omega E) \\ {\scriptstyle m(f)}\downarrow & & \downarrow{\scriptstyle C_*(\Omega f)} \\ TY & \xrightarrow{\simeq} & C_*(\Omega B) \end{array}$$

where the horizontal arrows are HAH quasi-isomorphisms. Then $m(f) : TX \to TY$ is a HAH morphism, equipped with a homotopy

$$h_{left} : \Delta_{TX} m(f) \approx (m(f) \otimes m(f))\Delta_{TY} \quad \textit{such that}$$

$$B(TY; TX) \xrightarrow{\simeq} B(C_*(\Omega B); C_*(\Omega E))$$

commutes with the diagonals up to homotopy. In particular, the algebra $H^*(F)$ is isomorphic to $H^*(\#B(TY; TX))$.

Proof. It is an immediate consequence of Property 8.11, followed by Theorem 9.4 and then by Proposition 6.7. $\boxed{QED}$

Remark 9.6 • The hypothesis on the compatibility of the homotopies of the cube in Theorem 9.4 is important. Indeed, in Corollary 9.5, even if $m(f) : TX \to TY$ commutes exactly with the diagonals, we cannot assume the homotopy h_{left}, given by Property 8.11, to be [0]. We will show it in Remark 10.5.

• But, if you take a bigger TY, you can have the exact commutation of $m(f)$ with the diagonals and you can assume h_{left} to be trivial. Indeed, we have the corollary:

Corollary 9.7 *Let $f : E \to B$ be a map between two pointed topological spaces with E path connected, F its homotopy fiber. We consider a homotopy commutative diagram of chain algebras as follows:*

$$\begin{array}{ccc} TX & \xrightarrow{\simeq} & C_*(\Omega E) \\ {\scriptstyle m(f)}\downarrow & & \downarrow{\scriptstyle C_*(\Omega f)} \\ TY & \xrightarrow{\simeq} & C_*(\Omega B) \end{array}$$

where

- *TX, TY are free DGA's and $TX \rightarrowtail TY$ is a cofibration*
- *$TX \overset{\simeq}{\to} C_*(\Omega E)$ is a HAH quasi-isomorphism.*

Then there is a diagonal on TY extending the diagonal of TX such that

1. *$TY \overset{\simeq}{\to} C_*(\Omega B)$ is a HAH quasi-isomorphism.*
2. *The coalgebra $H_*(TY \otimes_{TX} \mathbb{k})$ is isomorphic to $H_*(F)$.*

Proof. By Proposition 6.7, $C_*(F)$ is weakly DGC equivalent to $B(C_*\Omega B; C_*\Omega E)$. By Property 8.12, we obtain a diagonal on TY extending the diagonal of TX satisfying 1). By Theorem 9.4, the chain complexes morphism

$$B(TY; TX) \overset{\simeq}{\to} B(C_*\Omega B; C_*\Omega E)$$

commutes with the diagonals up to homotopy, where the diagonal on $B(TY; TX)$ is given by Corollary 6.2. By Property 5.5(i) and Property 6.3, the chain complexes morphism

$$B(TY; TX) \overset{\simeq}{\to} TY \otimes_{TX} \mathbb{k}$$

commutes with the diagonals. QED

10 The fiber of a suspension map

Lemma 10.1 *Let X be a path connected space. Then there is a natural DGH quasi-isomorphism $T\overline{C_*(X)} \overset{\simeq}{\to} C_*(\Omega\Sigma X)$.*

Proof. The adjunction map ad induces a morphism of coaugmented DGC's $C_*(ad) : C_*(X) \to C_*(\Omega\Sigma X)$. By universal property, the tensor algebra on the complex $\overline{C_*(X)}$, denoted $T\overline{C_*(X)}$, $C_*(ad)$ extends to a natural DGH morphism. By Theorem 1.4 appendix 2 [Hus], it is a quasi-isomorphism, since the functors H and T commute. $\boxed{QED}$

Theorem 10.2 *(i) Let $f : E \to B$ be a continuous map between path connected spaces. Then the coalgebra $C_*(F_{\Sigma f})$ is naturally weakly DGC equivalent to $B(T\overline{C_*(E)}; T\overline{C_*(B)})$.*

(ii) Consider a commutative diagram of coaugmented DGC's

$$\begin{array}{ccc} C & \xleftarrow{\simeq} & C_*(E) \\ \downarrow & & \downarrow{\scriptstyle C_*(f)} \\ C' & \xleftarrow{\simeq} & C_*(B) \end{array}$$

Then the coalgebra $C_*(F_{\Sigma f})$ is weakly DGC equivalent to $B(T\overline{C}; T\overline{C'})$.

Proof. It is a direct consequence of Lemma 10.1, Proposition 6.7 and Corollary 6.2. $\boxed{QED}$

Remark 10.3

- The category of chain connected DGC is certainly a proper model category. The dual functor # from the category of chain connected DGC to cochain connected DGA should preserve the proper model category structures. We don't know any reference in the literature about this, although some people told us that they have proved it (To prove it will lead us too far). Anyway, there exists a commutative diagram of coaugmented DGC as considered in Theorem 10.2(ii) such that C and C' are of finite type if $H^*(E)$ and $H^*(B)$ are also of finite type. So, here there is no problem of homotopy when we pass to models of the DGH $C_*(\Omega\Sigma X)$.

- If the diagram of Theorem 10.2(ii) commutes only up to chain DGC homotopy (Define a chain DGC homotopy to be a DGC morphism as in Definition 7.5), then we still have $H^*(\#B(T\overline{A(E)}; T\overline{A(B)})) \cong H^*(F_{\Sigma f})$ as algebras. Indeed the homotopy h of DGC's gives after tensorization, a homotopy of algebras $T(\overline{h})$ such that $\Delta T(\overline{h}) = (T(\overline{h}) \otimes T(\overline{h}))\Delta$. So we can apply Theorem 9.4.

Corollary 10.4 *Let $f : E \to B$ be a continuous map between path connected spaces such that $H_*(f)$ is injective. Then the DGC $TH_+(B) \otimes_{TH_+(E)} \mathbb{k}$ is isomorphic to $H_*(F_{\Sigma f})$.*

Proof. Since $H_*(f)$ is injective, we can apply Corollary 9.7 to the commutative diagram of DGA's:

$$\begin{array}{ccc} (TH_+(E), 0) & \xrightarrow{\simeq} & T\overline{C_*(E)} \\ {\scriptstyle TH_+(f)}\downarrow & & \downarrow{\scriptstyle T\overline{C_*(f)}} \\ (TH_+(B), 0) & \xrightarrow{\simeq} & T\overline{C_*(B)} \end{array}$$

Since the horizontal arrows induce the identity in homology, the diagonals on $TH_+(E)$ and $TH_+(B)$ must be obtained by tensorization of the diagonals of $H_+(E)$ and $H_+(B)$. QED

Remark 10.5 If $H_*(f)$ is not injective, Corollary 10.4 is not true: the algebra $H^*(F)$ does not depend only of $H_*(f)$. Indeed, we cannot assume the homotopy h_{left}, given by Property 8.11, to be [0], for the following homotopy commutative diagram of HAH's

$$\begin{array}{ccc} (TH_+(E), 0) & \xrightarrow{\simeq} & T\overline{C_*(E)} \\ {\scriptstyle TH_+(f)}\downarrow & & \downarrow{\scriptstyle T\overline{C_*(f)}} \\ (TH_+(B), 0) & \xrightarrow{\simeq} & T\overline{C_*(B)} \end{array}$$

For an example over $\mathbb{Z}_p$, we can take a map f from S^{2p-1} to $\mathbb{CP}^{p-1}$. Let y_2 be a generator of $H^2(F_{\Sigma f})$. If f is the Hopf map, there is a map $\psi : \mathbb{CP}^p \to F$ such that $H^2(\psi)$ is an isomorphism. So $y_2^p \neq 0$. If f is the constant map then $y_2^p = 0$.

Remark 10.6 When $H_*(B)$ is of finite type, we can prove and interpret topologically the isomorphism given by Corollary 10.4. Indeed, let ∂ : $\Omega\Sigma B \to F_{\Sigma f}$ be the connecting map. Since the Serre spectral sequence applied to

$$\Omega\Sigma E \stackrel{\Omega\Sigma f}{\to} \Omega\Sigma B \stackrel{\partial}{\to} F_{\Sigma f}$$

collapses, $H_*(\partial)$ is surjective and $\ker H_*(\partial)$ is isomorphic to $H_*(F_{\Sigma f}) \otimes H_+(\Omega\Sigma E)$. Now, by Theorem 1.4 appendix 2 [Hus], $\ker H_*(\partial)$ is the left ideal generated by the image of $H_*(E) \stackrel{H_*(f)}{\to} H_*(B) \stackrel{H_*(ad)}{\to} H_*(\Omega\Sigma B)$.

11 The category of augmented CDGA's as a category of cofibrant objects

The category of augmented $\mathbb{Q}$-CDGA's forms a cofibration category. We present a generalization of this context to augmented CDGA's over any field. This will allow us to link a particular CDGA of cohomology $H^*(F_f)$ with the fiber of any model of f in the proof of Theorem 12.2.

Definition 11.1 [Bro]([Bau1] I.1a.6) A *category of cofibrant objects* is a category with two classes of morphisms called *cofibrations* (denoted by $\rightarrowtail$) and *weak equivalences* (denoted by $\stackrel{\simeq}{\to}$) and an initial object Φ, which satisfies axioms $C1$, $C2$, $C3$ of Baues's definition of a cofibration category, and such that for every object X, $\Phi \rightarrowtail X$ is a cofibration (X is *cofibrant*).

Definition 11.2 Let IX be a cylinder for X, $f : X \to Y$ be a morphism. The *mapping cylinder* of f, denoted M_f, corresponding to IX is defined by the push out diagram:

$$\begin{array}{ccc} X \sqcup X & \xrightarrow{f \sqcup id} & Y \sqcup X \\ {\scriptstyle (i_0,i_1)}\downarrow & & \downarrow{\scriptstyle (j_0,j_1)} \\ IX & \longrightarrow & M_f \end{array}$$

Assume now that the category of cofibrant objects has also a final object, denoted by $*$. The *cone* of X, denoted by CX, is the mapping cylinder of $X \to *$.

Property 11.3 ([Bro] 1.1:Factorization lemma, [Bau1] I.1.8)
A factorization of f through a cofibration and a weak equivalence is given by $X \stackrel{j_1}{\rightarrowtail} M_f \stackrel{\simeq}{\to} Y$.

Definition 11.4 Let $A \rightarrowtail X$ be a cofibration. The *cofiber* X/A is defined by the following push out diagram. The *mapping cone* of f, denoted C_f, is defined either as the cofiber of $j_1 : X \rightarrowtail M_f$ or by the following push out diagram:

$$\begin{array}{ccccccc} A & \longrightarrow & X & & X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ * & \longrightarrow & X/A & & CX & \longrightarrow & C_f \end{array}$$

Property 11.5 The category of augmented CDGA's is a category of cofibrant objects, with $\Bbbk$ as initial and final object. The cofibrations are the morphisms $A \rightarrowtail B$ such that B is an A-semifree module. The weak equivalences are the quasi-isomorphisms.

Remark 11.6 Over a field of characteristic p, even if we restrict cofibrations to KS-extensions ([Bau1] I.8.5) (= relative Sullivan models ([FHT2] §14), the category of augmented CDGA's is not a cofibration category. Indeed, there is no lifting lemma ([FHT2] 14.6), but we have the following analogue of the relative Sullivan model:

Definition 11.7 Let B be an augmented DGA. An *augmented free B-algebra* is an inclusion of augmented DGA's: $(B, d) \rightarrowtail (B \otimes TV, D)$ such that $V = \oplus_{k \in \mathbb{N}} V(k)$ and $\forall k \in \mathbb{N}, d : V(k) \to B \otimes TV(< k)$. If B is a CDGA, then an augmented free B-algebra is really an augmented B-algebra.

Lemma 11.8 *(i) An augmented free B-algebra is B-semifree.*

(ii) Let $\alpha : B \to A$ be an augmented B-algebra. Then there is a quasi-isomorphism of augmented DGA's from an augmented free B-algebra to A extending α.

(iii) Suppose given a commutative diagram of augmented DGA's.

$$\begin{array}{ccc} B & \xrightarrow{\alpha} & A \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \eta}\ \simeq \\ B \otimes TV & \xrightarrow[\Psi]{} & C \end{array}$$

in which η is an A-algebra, α is a B-algebra, $B \otimes TV$ is a free B-algebra and η is a quasi-isomorphism. Then there is a morphism of augmented DGA's $\varphi : B \otimes TV \to A$ such that $\varphi \circ i = \alpha$ (φ extends α) and such that there is a homotopy of augmented DGA's $\eta \circ \varphi \approx \Psi$.

Corollary 11.9 *(i) Let $B \to A$ be a morphism of augmented CDGA's and $B \rightarrowtail B \otimes C \xrightarrow{\simeq} A$ a factorization of it through a cofibration and a weak equivalence. Then the DGA (not CDGA!) weak equivalence class of the cofiber C is unique.*

(ii) Suppose given a commutative diagram of augmented CDGA's

$$\begin{array}{ccc} B & \longrightarrow & A \\ \downarrow{\simeq} & & \downarrow{\simeq} \\ B' & \longrightarrow & A' \end{array}$$

Let $B \rightarrowtail B \otimes C \xrightarrow{\simeq} A$ (resp. $B' \rightarrowtail B' \otimes C' \xrightarrow{\simeq} A'$) be a factorization of $B \to A$ (resp. $B' \to A'$) through a cofibration and a weak equivalence. Then the cofibers C and C' are weakly DGA equivalent.

Proof.

(i) We apply the lifting lemma 11.8 to get a quasi-isomorphism $B \otimes TV \xrightarrow{\simeq} B \otimes C$ and we apply $\Bbbk \otimes_B -$ to have $TV \xrightarrow{\simeq} C$.

(ii) By Property 11.5, by push out, we obtain the factorization $B' \rightarrowtail B' \otimes C \xrightarrow{\simeq} A'$. Now by (i), $C \sim C'$. *QED*

12 The fiber of the model in the Anick range

Let $r \geq 1$ be a fixed integer. p is going to be the field's characteristic. We suppose now $p \neq 2$.

Definition 12.1 A topological space X is (r, p)-*mild* or in the *Anick range* if its homology is concentrated in degrees $> r$ and $\leq rp$.

Theorem 12.2 *Let $f : E \to B$ be a map between two topological spaces of finite type homology both (r, p)-mild with $H_{rp}(E) = 0$. There is a CDGA morphism, denoted $A(f) : A(B) \to A(E)$ such that*

1. *$C^*(X)$ is "naturally" weakly DGA equivalent to $A(X)$.*

2. *Let $A(B) \rightarrowtail A(B) \otimes D \xrightarrow{\simeq} A(E)$ be a factorization of $A(f) : A(B) \to A(E)$ such that $A(B) \otimes D$ is an $A(B)$-semifree module.*
   Then the cohomology algebra of the fiber, $H^*(F)$ is isomorphic to the cohomology of the cofiber, $H^*(D)$.

Remark 12.3 Over $\mathbb{Q}$, the functor A_{PL} verifies exactly that $C^*(X)$ is naturally weakly DGA equivalent to $A_{PL}(X)$ ([FHT2], 10.10). Here A is not a functor. But we denote A like a functor. So 1) just means that the usual diagram commutes for f.

Proof. • There is a commutative diagram of DGA's

$$\begin{array}{ccc} TX & \xrightarrow{\simeq} & C_*(\Omega E) \\ \downarrow & & \downarrow{\scriptstyle C_*(\Omega f)} \\ TY & \xrightarrow{\simeq} & C_*(\Omega B) \end{array}$$

where $TX \rightarrowtail TY$ is a cofibration and X and Y are graded vector spaces of finite type concentrated in degree $\geq r$ and $\leq rp - 1$. For example, we can take X to be $s^{-1}H_+(E)$ and Y to be $X \oplus sX \oplus s^{-1}H_+(B)$ (Apply the factorization lemma 11.3 with the mapping cylinder associated to the Baues-Lemaire cylinder) .

We apply Corollary 9.7 to have $H^*(\#B(TY; TX)) \cong H^*(F)$.
By the naturality of Anick's Theorem ([Maj] D.29 and D.21), we have a following commutative diagram of HAH's

$$\begin{array}{ccc} UL(E) & \xrightarrow{\cong} & TX \\ {\scriptstyle UL(f)}\downarrow & & \downarrow{\scriptstyle m(f)} \\ UL(B) & \xrightarrow{\cong} & TY \end{array}$$

with the horizontal arrows commuting with the diagonals up to natural homotopies.
We have for E and B the DGC quasi-isomorphisms $C_*L(X) \overset{\simeq}{\hookrightarrow} BUL(X) \overset{\simeq}{\to} BC_*(\Omega X)$.
By [FHT1] 6.3, $BC_*(\Omega X) \sim C_*(X)$ as DGC's.
So, by dualizing, we get $C^*L(X) \sim C^*(X)$, and we set $A(X) = C^*L(X)$. This proves 1.

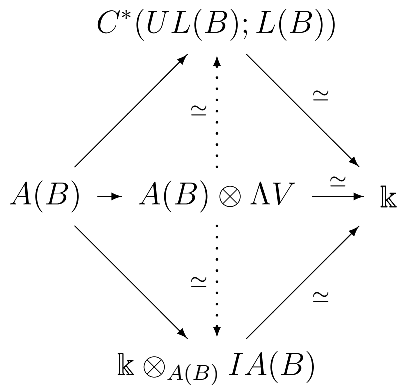
• By using just the naturality of the homotopies (Property 8.1) and the naturality of AW (Property 6.1), we get a weaker version of theorem 9.4 that we apply to get: $B(UL(B);UL(E)) \overset{\simeq}{\to} B(TY;TX)$ commutes up to homotopy with the diagonals.
We give $C_*(UL(B);L(E))$ the tensor product coalgebra structure of $UL(B)\otimes \Gamma sL(E)$.
The injection $C_*(UL(B);L(E)) \overset{\simeq}{\to} B(UL(B);UL(E))$ is a DGC quasi-isomorphism ([FHT1] 6.11).
So now $H^*(C^*(UL(B);L(E)) \cong H^*(F)$ as algebras.

• By Property 11.5 and by the factorization Lemma 11.3, we get the factorizations of $A(f)$ and of the augmentation of $A(B)$:

$$A(B) \rightarrowtail A(E) \otimes_{A(B)} IA(B) \overset{\simeq}{\to} A(E)$$

$$A(B) \rightarrowtail \Bbbk \otimes_{A(B)} IA(B) \overset{\simeq}{\to} \Bbbk$$

By [FHT1] 6.10, $C^*(UL(B);L(B))$ is a different cone from $\Bbbk \otimes_{A(B)} IA(B)$ and we can have another one: $A(B) \otimes \Lambda V$ such that $A(B) \rightarrowtail A(B) \otimes \Lambda V$ is a KS-extension.
So by the lifting lemma for surjection ([FHT2] 14.4) valid over any field, we have the commuting diagram:



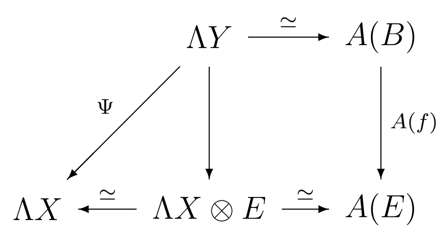
Now apply $A(E) \otimes_{A(B)} -$ and we have the quasi-isomorphisms:

$$A(E)\otimes_{A(B)}C^*(UL(B);L(B)) \overset{\simeq}{\leftarrow} A(E)\otimes_{A(B)}(A(B)\otimes\Lambda V) \overset{\simeq}{\rightarrow} A(E)\otimes_{A(B)}IA(B)$$

The two definitions of the mapping cone (Definition 11.4) give: $\Bbbk \otimes_{A(B)} (A(E) \otimes_{A(B)} IA(B)) \cong A(E) \otimes_{A(B)} (\Bbbk \otimes_{A(B)} IA(B))$ as CDGA's. By the universal property of push out, $C^*(UL(B); L(E)) \cong C^*(L(E)) \otimes_{C^*(L(B))} C^*(UL(B); L(B))$ as CDGA's. So now, we have the theorem for some factorization of $A(f)$. And we apply Corollary 11.9 to have it for any factorization.
QED

To construct a factorization of $A(f)$ is quite difficult. As in the rational case, we would rather construct a factorization of a model of $A(f)$:

Corollary 12.4 • *Let $A(f) : A(B) \to A(E)$ be a CDGA morphism as in Theorem 12.2.*
Let ΛY (resp. ΛX) be a Sullivan model of $A(B)$ (resp. $A(E)$). Then there is an acyclic CDGA E and there is a commutative diagram



• *Let $\Lambda Y \rightarrowtail \Lambda Y \otimes D \overset{\simeq}{\rightarrow} \Lambda X$ be a factorization of $\Psi : \Lambda Y \to \Lambda X$ such that $\Lambda Y \otimes D$ is a ΛY-semifree module.*
Then the algebra $H^*(F)$ is isomorphic to $H^*(D)$.

Proof. The first part of this corollary is just Proposition 7.7 and Remark 7.8 of [Hal2]. The second part is Corollary 11.9 and Theorem 12.2. *QED*

As in the rational case, we can take a factorization of Ψ with relative Sullivan models. But mod p, since the p^{th} power of an element of even degree is always a cycle, our relative Sullivan model will have infinitely many generators!
We'd rather use the divided power algebra ΓV where for $v \in V_{even}$, $v^p = 0$. But now arises the problem of constructing CDGA morphisms from a divided

power algebra to any CDGA where the p^{th} powers are not zero!
We give now an effective construction of a factorization of Ψ with divided power algebras. Let V and W be two graded vector spaces.

Definition 12.5 ([Hal2] 2.1) A Γ-*derivation* in $\Lambda V \otimes \Gamma W$ is a derivation D such that $D\gamma^k(w) = D(w)\gamma^{k-1}(w)$, $k \geq 1$, $v \in V^{even}$.

Property 12.6 ([Hal2] 2.1) Any linear map $V \oplus W \to \Lambda V \otimes \Gamma W$ of degree k extends to an unique Γ-derivation over $\Lambda V \otimes \Gamma W$.

Lemma 12.7 *Let $\Psi : (\Lambda Y, d) \to (\Lambda X, d)$ be a morphism between two minimal Sullivan models such that X and Y are concentrated in degree ≥ 2.*
Then we have a following factorization of Ψ through a cofibration and a surjective weak equivalence p:
$\Lambda Y, d \rightarrowtail \Lambda Y \otimes \Lambda coker\varphi \otimes \Gamma sker\varphi, D \underset{p}{\overset{\simeq}{\twoheadrightarrow}} \Lambda X, d$
where

- *φ is the composite $Y \hookrightarrow \Lambda Y \xrightarrow{\Psi} \Lambda X \twoheadrightarrow X$,*
- *D is a Γ-derivation,*
- *p is nul on $\Gamma sker\varphi$.*

Proof. By induction on the degree $n \in \mathbb{N}^*$. Suppose we have constructed the factorization:

$$\Lambda(Y^{\leq n}), d \rightarrowtail \Lambda(Y^{\leq n}) \otimes \Lambda(\mathrm{coker}\varphi^{\leq n}) \otimes \Gamma s(\ker\varphi^{\leq n}), D \underset{p_n}{\overset{\simeq}{\twoheadrightarrow}} \Lambda(X^{\leq n}), d$$

• Let $w \in \mathrm{coker}\varphi^{n+1}$. Define $p_{n+1}(w)$ obviously. $dp_{n+1}(w)$ is a cycle of $\Lambda X^{\leq n}$. Since p_n is a surjective quasi-isomorphism, there is a cycle $z \in \Lambda(Y^{\leq n}) \otimes \Lambda(\mathrm{coker}\varphi^{\leq n}) \otimes \Gamma s(\ker\varphi^{\leq n})$ such that $p_n(z) = dp_{n+1}(w)$. Define $Dw = z$.

• Let $v \in \ker\varphi^{n+1}$. Since p_{n+1} is a surjective algebra morphism, there is $v+u$ such that $u \in \Lambda^{\geq 2}(Y^{\leq n} \oplus \mathrm{coker}\varphi^{\leq n})$ and $p_{n+1}(v+u) = 0$. Since $D(v+u)$ is a cycle of $\Lambda(Y^{\leq n}) \otimes \Lambda(\mathrm{coker}\varphi^{\leq n}) \otimes \Gamma s(\ker\varphi^{\leq n})$ and p_n is a surjective quasi-isomorphism, there is $\gamma \in \Lambda Y^{\leq n} \otimes \mathrm{coker}\varphi^{\leq n} \otimes \Gamma s(\ker\varphi^{\leq n})$ such that $p_n(\gamma) = 0$ and $D\gamma = D(v+u)$. Define $Dsv = v + u - \gamma$.

• Now since $\Lambda Y^{n+1} \otimes \mathrm{coker}\varphi^{n+1} \otimes \Gamma s(\ker\varphi^{n+1}), \overline{D}$ is quasi-isomorphic to $\Lambda X^{n+1}, 0$ and since p_n is a quasi-isomorphism, by comparison of the E_2-term of the algebraic Serre spectral sequence, p_{n+1} is a quasi-isomorphism. *QED*

Example 12.8 Let $[f]$ be a generator of $\pi_2(\mathbb{CP}^n)$ with $n \geq 2$.
ψ is $(\Lambda(x_2, y_{2n+1}), d) \to (\Lambda(x_2, z_3), d)$ with $dy_{2n+1} = x_2^{n+1}$ and $dz_3 = x_2^2$.
ψ factorises through the CDGA $(\Lambda(x_2, y_{2n+1}, z_3) \otimes \Gamma s y_{2n+1}, D)$ with $Dz_3 = x_2^2$ and $Dsy_{2n+1} = y_{2n+1} - z_3 x_2^{n-1}$. So $H^*(F) \cong \Lambda z_3 \otimes \Gamma s y_{2n+1}$ for $p \geq 2n$.

Corollary 12.9 *[Hal2] If X is (r,p)-mild then the algebra $H^*(\Omega X)$ is isomorphic to ΓsV where ΛV is a minimal Sullivan model of $A(X)$.*

Remark 12.10 The hypotheses of the Theorem 12.2 are necessary:

• B must be (r,p)-mild. Indeed even for a path fibration $\Omega X \to PX \to X$, a commutative model of X does not determine the cohomology algebra of the loop space. $\Sigma\mathbb{CP}^p$ and $S^3 \vee .. \vee S^{2p+1}$ just not $(2,p)$-mild, have a same commutative model but the cohomology algebras of their loop spaces are not isomorphic.

• E and B both (r,p)-mild is not enough. $H_{rp}(E)$ must also be zero. Take the same example as in Remark 10.5: the suspension of the Hopf map $\Sigma\eta : \Sigma S^{2p-1} \to \Sigma\mathbb{CP}^{p-1}$.

Remark 12.11 Over $\mathbb{Q}$, replacing A by A_{PL}, the Grivel-Thomas-Halperin theorem "the fiber of a model is a model of the fiber" asserts that $D \sim C^*(F)$ as DGA's. But over a fields of characteristic p, we can't improve the theorem, by $D \sim C^*(F)$ as DGA's. Indeed, let X the $2p+3$ skeleton of a $K(\mathbb{Z}, 4)$. X is $(3,p)$-mild and $C^*\Omega X$ has no commutative model.
Proof. A consequence of Milnor is that there exist two CW-complexes denoted Y and $K(\mathbb{Z}, 3)$ with the same $2p+2$ skeletons, resp. homotopic to ΩX and $\Omega K(\mathbb{Z}, 4)$. The two morphisms of topological monoids

$$\Omega(Y^{(2p+2)}) \to \Omega B \quad \text{and} \quad \Omega(K(\mathbb{Z},3)^{(2p+2)}) \to \Omega K(\mathbb{Z},3)$$

induce in homology two algebra morphisms which are isomorphism in degree $\leq 2p$. $H_*(\Omega K(\mathbb{Z},3)) \cong \Gamma\alpha_2$ as algebras. So ΩB is 1-connected, $H_2(\Omega B) = \mathbb{Z}_p\alpha_2$ and $\alpha_2^p = 0$. Suppose $C^*(Y)$ has a commutative model. Then we have a quasi-isomorphism of algebras $U(L,d) \xrightarrow{\simeq} C_*(\Omega B)$. The homology of an universal enveloping algebra of a DGL $U(L,d)$ is an universal enveloping algebra of a Lie algebra UE ([Hal2] 8.3). So $H_*(\Omega B)$ admits by the Poincaré-Birkoff-Witt Theorem a basis containing $\alpha_2^p \neq 0$. QED

Over $\mathbb{Q}$, by ([Bau2] 4.8), there is a quasi-isomorphism of DGH's (and not only of HAH's) $UL \xrightarrow{\simeq} C_*\Omega X$. Over $\mathbb{Z}_p$, this counterexample proves that we can't get rid of the homotopy!

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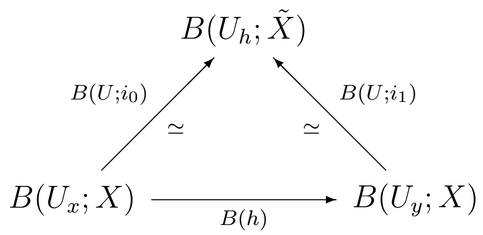
Troisième partie
Annexes

13 Détails des preuves de 8 et 9

Justification de la définition 8.2

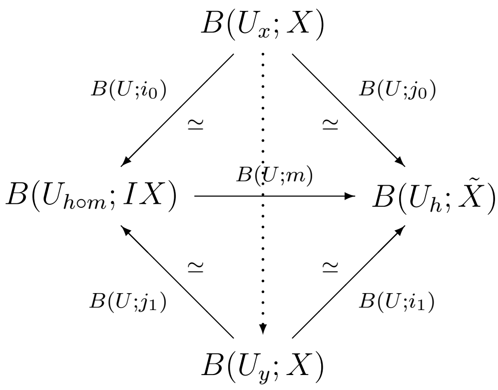
Comme le foncteur B préserve les quasi-isomorphismes ([FHT1] 4.3(iii)), $B(U;i_0)$ et $B(U;i_1)$ sont des quasi-isomorphismes de complexes de chaînes, donc des équivalences d'homotopies de chaînes. Les morphismes de complexes de chaînes $B(h)$ sont donc bien déterminés de manière unique à homotopie près par le diagramme commutatif à homotopie près suivant :



QED

Preuve de la propriété 8.3

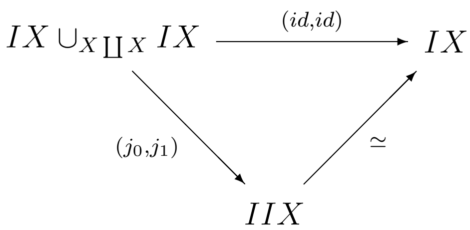
Par fonctorialité et d'après la propriété 7.6, nous avons le diagramme commutatif

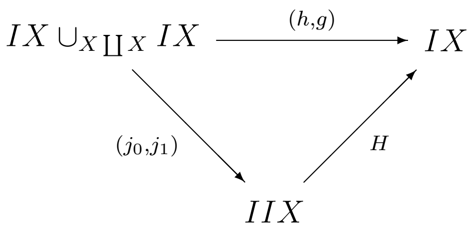


Donc par définition, $B(h)$ et $B(h \circ m)$ sont homotopes. QED

Preuve de la propriété 8.4

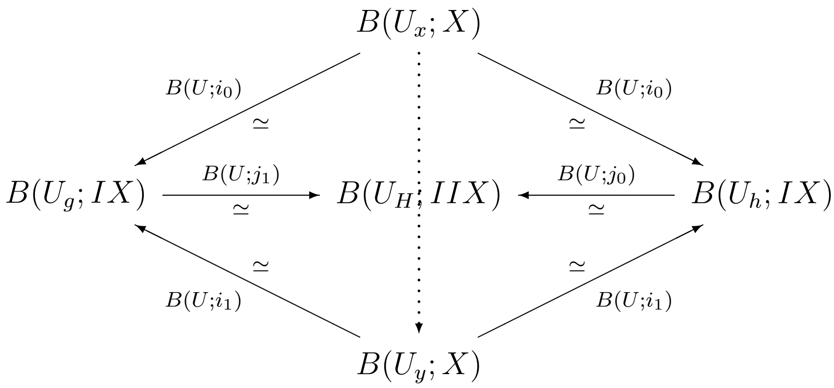
• Nous démontrons que $[B(h)]$ ne dépend que de $[h]$. Soit h, $g : x \approx y$ deux homotopies de x vers y partant du cylindre IX. D'après la Définition 7.10, si h est homotope à g, alors nous avons les diagrammes commutatifs





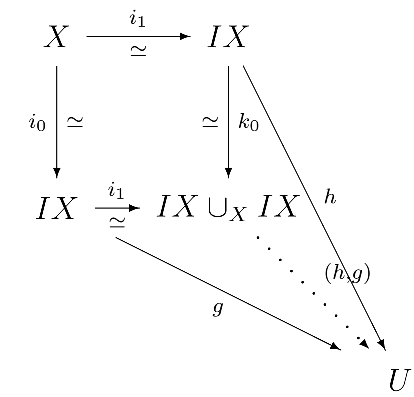
où $\quad j_0 \circ i_0 = j_1 \circ i_0 \quad$ et $\quad j_0 \circ i_1 = j_1 \circ i_1$.

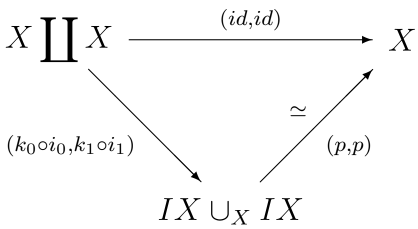
Nous avons donc le diagramme commutatif



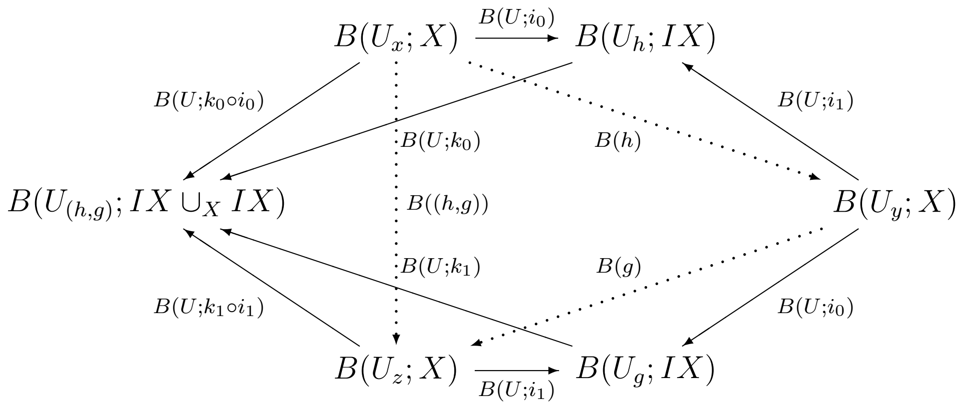
Donc $B(h)$ est homotope à $B(g)$.

• Nous démontrons que $B(h + g)$ est homotope à $B(g) \circ B(h)$. Par définition de la somme d'homotopies (Définition 7.8 (ii)), nous avons les diagrammes commutatifs





Nous avons donc le diagramme commutatif



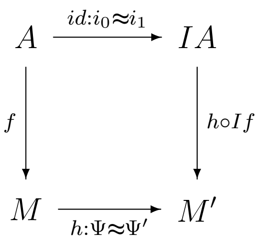
Donc $B((h,g))$ est homotope à $B(g) \circ B(h)$. Par la propriété 8.3,

$$B(h+g) = B((h,g) \circ m) \approx B((h,g))$$

QED

Preuve du lemme 8.5

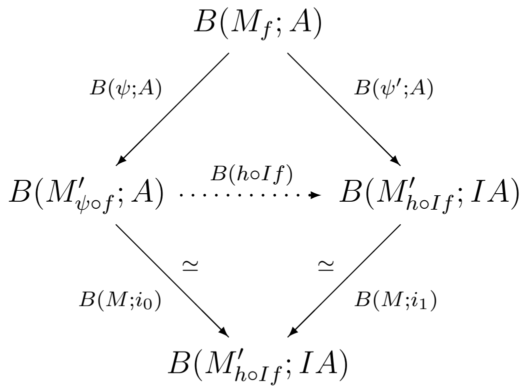
- (i) Nous pouvons appliquer la propriété 8.1 au "diagramme"



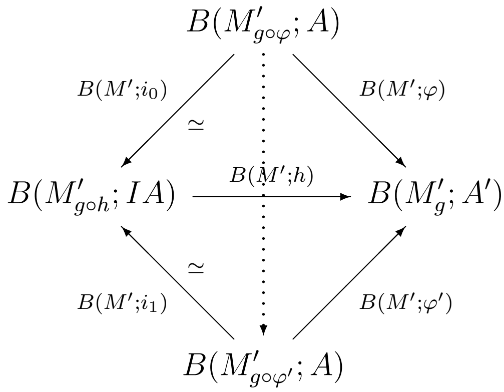
En effet, la condition de naturalité est vérifiée :

$$h \circ If = h \circ If \circ id$$

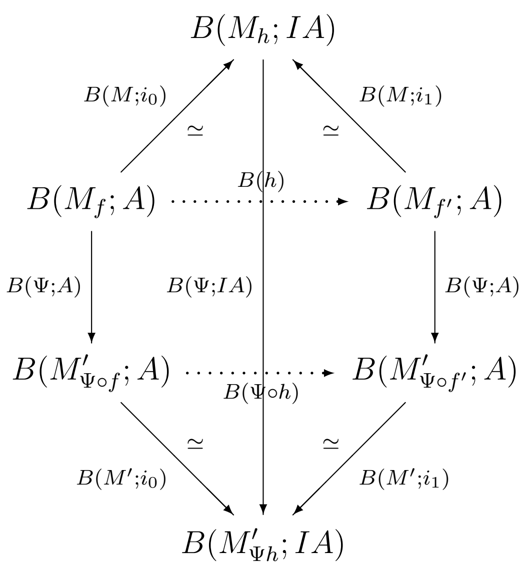
Donc nous avons le diagramme commutatif



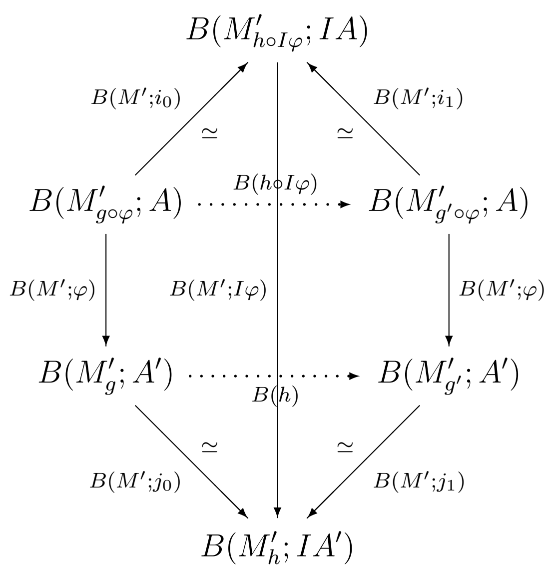
• (ii) découle du diagramme commutatif



• (iii) découle du diagramme commutatif



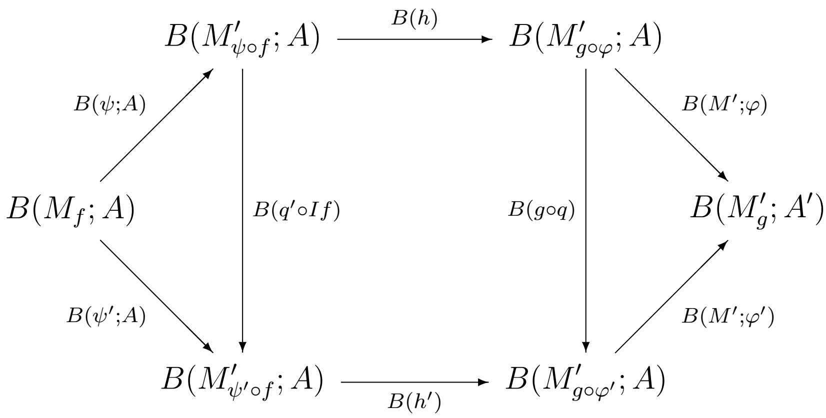
• (iv) découle du diagramme commutatif



QED

Preuve de la proposition 8.7

Par la propriété 8.4 et le Lemme 8.5 (i) et (ii), nous obtenons le diagramme commutatif à homotopie près



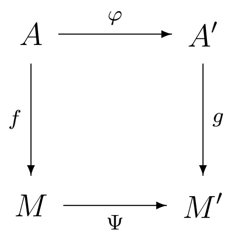
Donc par définition, (Définition 8.6)

$$B(\psi; \varphi; h) \approx B(\psi'; \varphi'; h')$$

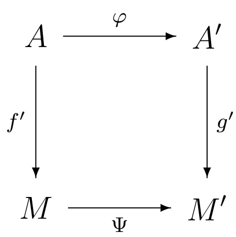
QED

Pour démontrer la proposition 8.8, nous avons besoin de la proposition

Proposition 13.1 *Soit*



un diagramme commutatif à l'homotopie près $h : \Psi \circ f \approx g \circ \varphi$. *Soient* $q : f \approx f'$ *et* $q' : g \approx g'$ *deux homotopies. Alors*



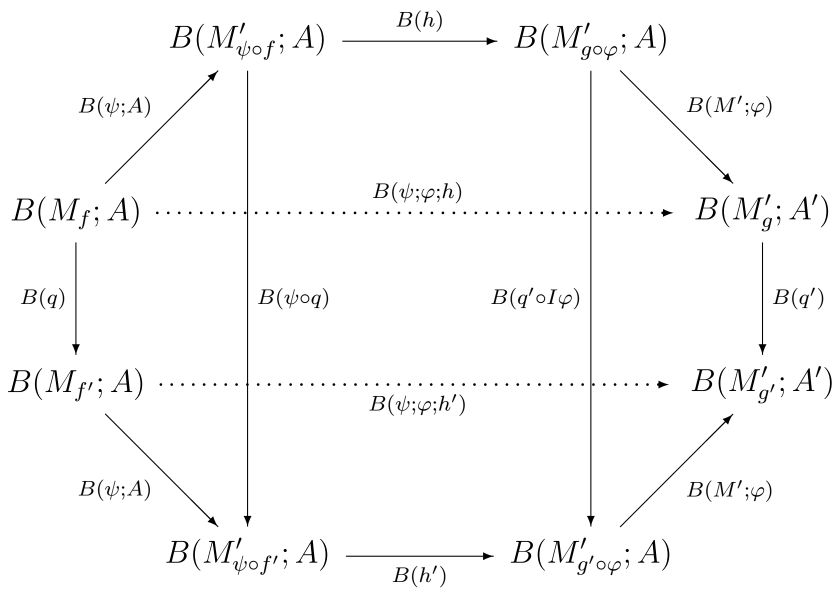
est un diagramme qui commute à l'homotopie près $h' = -\psi q + h + q' I\varphi$, *et les deux composés*

$$B(M_f;A) \xrightarrow{B(\psi;\varphi;h)} B(M'_g;A') \xrightarrow{B(q')} B(M'_{g'};A') \quad et$$

$$B(M_f;A) \xrightarrow{B(q)} B(M_{f'};A) \xrightarrow{B(\psi;\varphi;h')} B(M'_{g'};A')$$

sont homotopes.

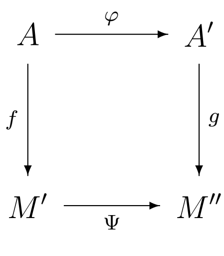
Preuve. Par transitivité, $\Psi f' \approx \Psi f \approx g\varphi \approx g'\varphi$. Par la propriété 8.4 et le Lemme 8.5 (iii) et (iv), nous avons le diagramme commutatif



QED

Preuve de la proposition 8.8

Nous appliquons la proposition 13.1 au diagramme commutatif



et aux homotopies $-h : f'\circ\varphi \approx \psi\circ f$ et $h' : \psi'\circ f' \approx f''\circ\varphi'$. Nous obtenons alors le diagramme commutatif à homotopie près.

$$\begin{array}{ccc}
B(M_f;A) & \xrightarrow{B(\psi'\circ\psi;\varphi'\circ\varphi;h'')} & B''(M_{f''};A'') \\
\Big\downarrow{\scriptstyle B(\psi;A)} & & \Big\uparrow{\scriptstyle B(M'';\varphi')} \\
B(M'_{\psi\circ f};A) & \xrightarrow{B(\psi';\varphi;h'')} & B(M''_{f''\circ\varphi'};A') \\
\Big\uparrow{\scriptstyle B(-h)} & & \Big\uparrow{\scriptstyle B(h')} \\
B(M'_{f'\circ\varphi};A) & \xrightarrow{B(\psi';\varphi;0)} & B(M''_{\psi'\circ f};A') \\
& {\scriptstyle B(M';\varphi)}\searrow \quad \nearrow{\scriptstyle B(\psi;A')} & \\
& B(M'_{f'};A') &
\end{array}$$

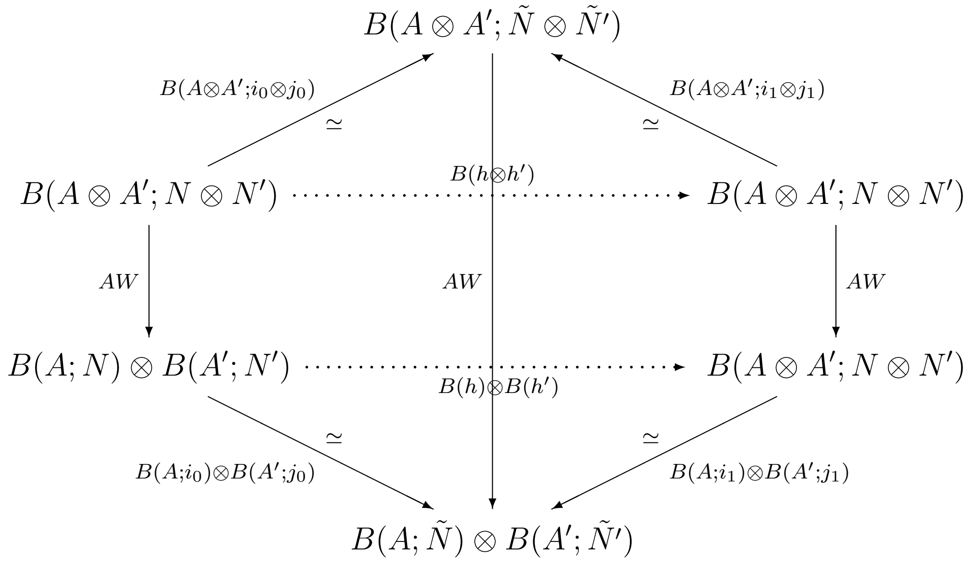
QED

Preuve de la proposition 9.3

Supposons que l'homotopie h (resp. h') parte de l'objet d'homotopie gauche $\tilde{A}$ (resp. $\tilde{A}'$). Soient i_0, i_1 (resp. j_0, j_1) les deux inclusions de A dans $\tilde{A}$ (resp. de A' dans $\tilde{A}'$). Alors (Propriété 9.1) on a le diagramme commutatif

$$\begin{array}{ccc}
A\otimes A' \coprod A\otimes A' & \xrightarrow{(\psi\circ f\otimes\psi'\circ f',\varphi\circ g\otimes\varphi'\circ g')} & N\otimes N') \\
& {\scriptstyle (i_0\otimes j_0,i_1\otimes j_1)}\searrow \quad \nearrow{\scriptstyle h\otimes h'} & \\
& \tilde{A}\otimes\tilde{A}' &
\end{array}$$

Donc par naturalité de l'application d'Alexander-Whitney (Propriété 6.1), nous obtenons le diagramme commutatif



$$\text{Donc} \quad (B(h) \otimes B(h')) \circ AW \approx AW \circ B(h \otimes h').$$

A nouveau par naturalité de AW, et par définition des $B(\Psi; \varphi; h)$s (Définition 8.6) nous obtenons

$$(B(\Psi; \varphi; h) \otimes B(\Psi'; \varphi'; h')) \circ AW \approx AW \circ B(\Psi \otimes \Psi'; \varphi \otimes \varphi'; h \otimes h').$$

QED

14 Preuve du lemme 11.8(iii)

Pour démontrer le lemme 11.8(iii), nous utilisons la propriété et les deux lemmes suivants. Soit B une ADG (augmentée).

Property 14.1 (Propriété universelle des B-algèbres libres (augmentées)) Soit $\alpha : B \to A$ une B-algèbre (augmentée). Soit $\varphi : V \to A$ un morphisme de complexes (augmentés). Alors il existe un unique morphisme de ADG (augmentées) $B \otimes TV \to A$ (resp. $B \otimes T\overline{V} \to A$) étendant φ et α.

Lemma 14.2 *(Lemme de relèvement avec surjection) Soit*

$$\begin{array}{ccc} B & \xrightarrow{\ \alpha\ } & A \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle \eta\ \simeq} \\ B \otimes T\overline{V} & \xrightarrow[\ \Psi\ ]{} & C \end{array}$$

un diagramme commutatif de ADG (augmentées) tel que α soit une B-algèbre, $B \otimes TV$ une B-algèbre libre et η un quasi-isomorphisme surjectif. Alors il existe un morphisme de ADG (augmentées) $\varphi : B \otimes TV \to A$ tel que $\varphi \circ i = \Psi$ (φ étend α) et $\eta \circ \varphi = \Psi$.

Lemma 14.3 *(factorisation en un quasi-isomorphism suivi d'une surjection) Soit $\eta : A \to C$ un morphisme de B-algèbres (augmentées) tel que C soit une A-algèbre. Alors on obtient un diagramme commutatif de B-algèbres (augmentées)*

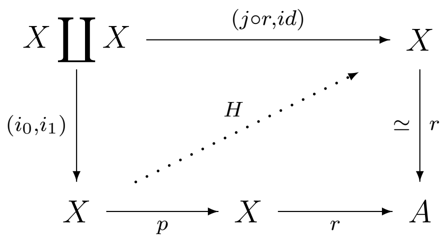
$$\begin{array}{ccc} A & \xrightarrow{\quad \eta \quad} & C \\ & {\scriptstyle j}\searrow^{\simeq} \quad \nearrow{\scriptstyle q} & \\ & X & \end{array}$$

où q est surjectif et j admet un rétracte $r : X \to A$ dans la catégorie des B-algèbres (augmentées). Donc il existe une homotopie de ADG (augmentées) entre id et $j \circ r$.

Preuve. Nous allons démontrer le lemme en prenant $X = A \otimes T(C \oplus dC)$ (resp. $X = A \otimes T(\overline{C} \oplus d\overline{C})$.

• Nous appliquons la propriété universelle des B-algèbres libres (augmentées) à $C \oplus dC \twoheadrightarrow C$ (resp. $\Bbbk \oplus \overline{C} \oplus d\overline{C} \twoheadrightarrow C$) et à la A-algèbre C et nous vérifions que $j : A \overset{\simeq}{\hookrightarrow} X$ et $q : X \twoheadrightarrow A$ sont des morphismes de B-algèbres.

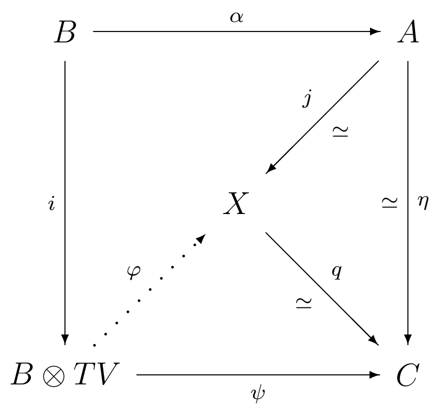
• L'homotopie de ADG (augmentées) H est obtenue par le lemme de relèvement avec surjection dans le diagramme commutatif



où IX est un cylindre. $\boxed{QED}$

Preuve du lemme 11.8(iii)

Nous appliquons le lemme de factorisation en un quasi-isomorphisme suivi d'une surjection et le lemme de relèvement avec surjection au diagramme commutatif



Nous obtenons $r \circ \varphi \circ i = \alpha$ et $\eta \circ r \circ \varphi = q \circ j \circ r \circ \varphi \approx q \circ \varphi = \psi$. $\boxed{QED}$

Résumé

La cohomologie d'un espace topologique à coefficients dans un corps est un invariant homotopique important en Mathématiques. Sa structure d'espace vectoriel peut se calculer par de nombreux moyens. Par contre, sa structure d'algèbre est plus difficile à calculer.

Souvent, un espace topologique intervient dans une fibration. Considèrons une fibration p de fibre F. Une question fondamentale est de savoir quelles sont les données algébriques sur p, qui à la fois déterminent et permettent de calculer l'algèbre de cohomologie de F.

Félix, Halperin et Thomas ont prouvé que cette algèbre est déterminée par un morphisme d'algèbres de Hopf induit par p au niveau des complexes de chaînes singulières sur les espaces de lacets, grâce à la bar construction.

Malheureusement, cela ne permet pas de calculer cette algèbre de cohomologie, car on ne peut pas passer aux modèles dans la catégorie des algèbres de Hopf. Par contre, dans la catégorie des algèbres de Hopf à homotopie près, introduite par Anick, il est possible de passer aux modèles.

Nous généralisons la bar construction de Félix, Halperin et Thomas aux morphismes d'algèbres de Hopf à homotopie près et établissons une condition de compatibilité des homotopies, pour que cette bar construction donne toujours l'algèbre de cohomologie de F.

Cela nous permet de donner une méthode pour calculer cette algèbre pour une fibration p obtenue par suspension. L'application la plus frappante est une généralisation à coefficients dans un corps de caractéristique différente de deux et dans le domaine d'Anick, d'un théorème classique de l'homotopie rationnelle affirmant que "la fibre du modèle est un modèle de la fibre".