

# Rational homotopy – Sullivan models

*Luc Menichi*

*Faculte des Sciences*

*2 Boulevard Lavoisier, 49045 Angers Cedex 01, France*

*email: luc.menichi@univ-angers.fr*

**Abstract.** This chapter is a short introduction to Sullivan models. In particular, we find the Sullivan model of a free loop space and use it to prove the Vigué-Poirrier-Sullivan theorem on the Betti numbers of a free loop space.

In the previous chapter, we have seen the following theorem due to Gromoll and Meyer.

**Theorem 0.1.** *Let  $M$  be a compact simply connected manifold. If the sequence of Betti numbers of the free loop space on  $M$ ,  $M^{S^1}$ , is unbounded then any Riemannian metric on  $M$  carries infinitely many non trivial and geometrically distinct closed geodesics.*

In this chapter, using Rational homotopy, we will see exactly when the sequence of Betti numbers of  $M^{S^1}$  over a field of characteristic 0 is bounded (See Theorem 6.1 and its converse Proposition 5.5). This was one of the first major applications of rational homotopy.

Rational homotopy associates to any rational simply connected space, a commutative differential graded algebra. If we restrict to almost free commutative differential graded algebras, that is "Sullivan models", this association is unique.

## 1 Graded differential algebra

### 1.1 Definition and elementary properties

All the vector spaces are over  $\mathbb{Q}$  (or more generally over a field  $\mathbf{k}$  of characteristic 0). We will denote by  $\mathbb{N}$  the set of non-negative integers.

**Definition 1.1.** A (non-negatively upper) *graded vector space*  $V$  is a family  $\{V^n\}_{n \in \mathbb{N}}$  of vector spaces. An element  $v \in V_i$  is an element of  $V$  of *degree*  $i$ . The degree of  $v$  is denoted  $|v|$ . A *differential*  $d$  in  $V$  is a sequence of linear maps

$d^n : V^n \rightarrow V^{n+1}$  such that  $d^{n+1} \circ d^n = 0$ , for all  $n \in \mathbb{N}$ . A differential graded vector space or *complex* is a graded vector space equipped with a differential. A morphism of complexes  $f : V \xrightarrow{\sim} W$  is a *quasi-isomorphism* if the induced map in homology  $H(f) : H(V) \xrightarrow{\cong} H(W)$  is an isomorphism in all degrees.

**Definition 1.2.** A *graded algebra* is a graded vector space  $A = \{A^n\}_{n \in \mathbb{N}}$ , equipped with a multiplication  $\mu : A^p \otimes A^q \rightarrow A^{p+q}$ . The algebra is *commutative* if  $ab = (-1)^{|a||b|}ba$  for all  $a$  and  $b \in A$ .

**Definition 1.3.** A differential graded algebra or *dga* is a graded algebra equipped with a differential  $d : A^n \rightarrow A^{n+1}$  which is also a *derivation*: this means that for  $a$  and  $b \in A$

$$d(ab) = (da)b + (-1)^{|a|}a(db).$$

A *cdga* is a commutative dga.

**Example 1.4.** 1) Let  $(B, d_B)$  and  $(C, d_C)$  be two cdgas. Then the tensor product  $B \otimes C$  equipped with the multiplication

$$(b \otimes c)(b' \otimes c') := (-1)^{|c||b'|}bb' \otimes cc'$$

and the differential

$$d(b \otimes c) = (db) \otimes c + (-1)^{|b|}b \otimes dc.$$

is a cdga. The *tensor product of cdgas* is the sum (or coproduct) in the category of cdgas.

2) More generally, let  $f : A \rightarrow B$  and  $g : A \rightarrow C$  be two morphisms of cdgas. Let  $B \otimes_A C$  be the quotient of  $B \otimes C$  by the sub graded vector spanned by elements of the form  $bf(a) \otimes c - b \otimes g(a)c$ ,  $a \in A$ ,  $b \in B$  and  $c \in C$ . Then  $B \otimes_A C$  is a cdga such that the quotient map  $B \otimes C \rightarrow B \otimes_A C$  is a morphism of cdgas. The cdga  $B \otimes_A C$  is the pushout of  $f$  and  $g$  in the category of cdgas:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow \\ C & \longrightarrow & B \otimes_A C \\ & \searrow & \downarrow \exists! \\ & & D \end{array}$$

3) Let  $V$  and  $W$  be two graded vector spaces. We denote by  $\Lambda V$  the free graded commutative algebra on  $V$ .

If  $V = \mathbb{Q}v$ , i. e. is of dimension 1 and generated by a single element  $v$ , then

- $\Lambda V$  is  $E(v) = \mathbb{Q} \oplus \mathbb{Q}v$ , the exterior algebra on  $v$  if the degree of  $v$  is odd and

- $\Lambda V$  is  $\mathbb{Q}[v] = \bigoplus_{n \in \mathbb{N}} \mathbb{Q}v^n$ , the polynomial or symmetric algebra on  $v$  if the degree of  $v$  is even.

Since  $\Lambda$  is left adjoint to the forgetful functor from the category of commutative graded algebras to the category of graded vector spaces,  $\Lambda$  preserves sums: there

is a natural isomorphism of commutative graded algebras  $\Lambda(V \oplus W) \cong \Lambda V \otimes \Lambda W$ .

Therefore  $\Lambda V$  is the tensor product  $E(V^{odd}) \otimes S(V^{even})$  of the exterior algebra on the generators of odd degree and of the polynomial algebra on the generators of even degree.

**Definition 1.5.** Let  $f : A \rightarrow B$  be a morphism of commutative graded algebras. Let  $d : A \rightarrow B$  be a linear map of degree  $k$ . By definition,  $d$  is a  $(f, f)$ -derivation if for  $a$  and  $b \in A$

$$d(ab) = (da)f(b) + (-1)^{k|a|}f(a)(db).$$

**Property 1.6** (Universal properties). 1) Let  $i_B : B \hookrightarrow B \otimes \Lambda V$ ,  $b \mapsto b \otimes 1$  and  $i_V : V \hookrightarrow B \otimes \Lambda V$ ,  $v \mapsto 1 \otimes v$  be the inclusion maps. Let  $\varphi : B \rightarrow C$  be a morphism of commutative graded algebras. Let  $f : V \rightarrow C$  be a morphism of graded vector spaces. Then  $\varphi$  and  $f$  extend uniquely to a morphism  $B \otimes \Lambda V \rightarrow C$  of commutative graded algebras such that the following diagram commutes

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & C & \xleftarrow{f} & V \\ & \searrow i_B & \uparrow \exists! & \swarrow i_V & \\ & & B \otimes \Lambda V & & \end{array}$$

2) Let  $d_B : B \rightarrow B$  be a derivation of degree  $k$ . Let  $d_V : V \rightarrow B \otimes \Lambda V$  be a linear map of degree  $k$ . Then there is a unique derivation  $d$  such that the following diagram commutes.

$$\begin{array}{ccccc} B & \xrightarrow{i_B} & B \otimes \Lambda V & \xleftarrow{d_V} & V \\ d_B \uparrow & & \uparrow \exists! d & \swarrow i_V & \\ B & \xrightarrow{i_B} & B \otimes \Lambda V & & \end{array}$$

3) Let  $f : \Lambda V \rightarrow B$  be a morphism of commutative graded algebras. Let  $d_V : V \rightarrow B$  be a linear map of degree  $k$ . Then there exists a unique  $(f, f)$ -derivation  $d$  extending  $d_V$ :

$$\begin{array}{ccc} V & \xrightarrow{d_V} & B \\ i_V \downarrow & \nearrow \exists! d & \\ & & \Lambda V \end{array}$$

*Proof.* 1) Since  $\Lambda V$  is the free commutative graded algebra on  $V$ ,  $f$  can be extended to a morphism of graded algebras  $\Lambda V \rightarrow C$ . Since the tensor product of commutative graded algebras is the sum in the category of commutative graded algebras, we obtain a morphism of commutative graded algebras from  $B \otimes \Lambda V$  to  $C$ .

2) Since  $b \otimes v_1 \dots v_n$  is the product  $(b \otimes 1)(1 \otimes v_1) \dots (1 \otimes v_n)$ ,  $d(b \otimes v_1 \dots v_n)$

is given by

$$d_B(b) \otimes v_1 \dots v_n + \sum_{i=1}^n (-1)^{k(|b|+|v_1|+\dots+|v_{i-1}|)} (b \otimes v_1 \dots v_{i-1}) (d_V v_i) (1 \otimes v_{i+1} \dots v_n)$$

3) Similarly,  $d(v_1 \dots v_n)$  is given by

$$\sum_{i=1}^n (-1)^{k(|v_1|+\dots+|v_{i-1}|)} f(v_1) \dots f(v_{i-1}) d_V(v_i) f(v_{i+1}) \dots f(v_n)$$

□

## 1.2 Sullivan models of spheres

**Sullivan models of odd spheres**  $S^{2n+1}$ ,  $n \geq 0$ .

Consider a cdga  $A(S^{2n+1})$  whose cohomology is isomorphic as graded algebras to the cohomology of  $S^{2n+1}$  with coefficients in  $\mathbf{k}$ :

$$H^*(A(S^{2n+1})) \cong H^*(S^{2n+1}).$$

When  $\mathbf{k}$  is  $\mathbb{R}$ , you can think of  $A$  as the De Rham algebra of forms on  $S^{2n+1}$ . There exists a cycle  $v$  of degree  $2n + 1$  in  $A(S^{2n+1})$  such that

$$H^*(A(S^{2n+1})) = \Lambda[v].$$

The inclusion of complexes  $(\mathbf{k}v, 0) \hookrightarrow A(S^{2n+1})$  extends to a unique morphism of cdgas  $m : (\Lambda v, 0) \rightarrow A(S^{2n+1})$  (Property 1.6):

$$\begin{array}{ccc} (\mathbf{k}v, 0) & \longrightarrow & A(S^{2n+1}) \\ \downarrow & \nearrow \exists! m & \\ (\Lambda v, 0) & & \end{array}$$

The induced morphism in homology  $H(m)$  is an isomorphism. We say that  $m : (\Lambda v, 0) \xrightarrow{\cong} A(S^{2n+1})$  is a Sullivan model of  $S^{2n+1}$ .

**Sullivan models of even spheres**  $S^{2n}$ ,  $n \geq 1$ .

Exactly as above, we construct a morphism of cdga  $m_1 : (\Lambda v, 0) \rightarrow A(S^{2n})$ . But now,  $H(m_1)$  is not an isomorphism:

$H(m_1)(v) = [v]$ . Therefore  $H(m_1)(v^2) = [v^2] = [v]^2 = 0$ . Since  $[v^2] = 0$  in  $H^*(A(S^{2n}))$ , there exists an element  $\psi \in A(S^{2n})$  of degree  $4n - 1$  such that  $d\psi = v^2$ .

Let  $w$  denote another element of degree  $4n - 1$ . The morphism of graded vector spaces  $\mathbf{k}v \oplus \mathbf{k}w \hookrightarrow A(S^{2n})$ , mapping  $v$  to  $v$  and  $w$  to  $\psi$  extends to a unique morphism of commutative graded algebras  $m : \Lambda(v, w) \rightarrow A(S^{2n})$  (1) of

Property 1.6):

$$\begin{array}{ccc} \mathbf{k}v \oplus \mathbf{k}w & \longrightarrow & A(S^{2n}) \\ \downarrow & \nearrow \exists! m & \\ \Lambda(v, w) & & \end{array}$$

The linear map of degree +1,  $d_V : V := \mathbf{k}v \oplus \mathbf{k}w \rightarrow \Lambda(v, w)$  mapping  $v$  to 0 and  $w$  to  $v^2$  extends to a unique derivation  $d : \Lambda(v, w) \rightarrow \Lambda(v, w)$  (2) of Property 1.6).

$$\begin{array}{ccc} \mathbf{k}v \oplus \mathbf{k}w & \xrightarrow{d_V} & \Lambda(v, w) \\ \downarrow & \nearrow \exists! d & \\ \Lambda(v, w) & & \end{array}$$

Since  $d$  is a derivation of odd degree,  $d \circ d$  (which is equal to  $1/2[d, d]$ ) is again a derivation. The following diagram commutes

$$\begin{array}{ccccc} V & \xrightarrow{d_V} & \Lambda V & \xrightarrow{d} & \Lambda V \\ \downarrow & \nearrow d & \nearrow d \circ d & & \\ \Lambda V & & & & \end{array}$$

Since the composite  $d \circ d_V$  is null, by unicity (2) of Property 1.6), the derivation  $d \circ d$  is also null. Therefore  $(\Lambda V, d)$  is a cdga. This is the general method to check that  $d \circ d = 0$ .

Denote by  $d_A$  the differential on  $A(S^{2n})$ . Let's check now that  $d_A \circ m = m \circ d$ . Since  $d_A$  and  $d$  are both  $(id, id)$ -derivations,  $d_A \circ m$  and  $m \circ d$  are both  $(m, m)$ -derivations.

Since  $d_A(m(v)) = d_A(v) = 0 = m(0) = m(d(v))$  and  $d_A(m(w)) = d_A(\psi) = v^2 = m(v^2) = m(d(w))$ ,  $d_A \circ m$  and  $m \circ d$  coincide on  $V$ . Therefore by unicity (3) of Property 1.6),  $d_A \circ m = m \circ d$ . Again, this method is general. So finally, we have proved that  $m$  is a morphism of cdgas. Now we prove that  $H(m)$  is an isomorphism, by checking that  $H(m)$  sends a basis to a basis.

## 2 Sullivan models

### 2.1 Definitions

Let  $V$  be a graded vector space. Denote by  $V^+ = V^{\geq 1}$  the sub graded vector space of  $V$  formed by the elements of  $V$  of positive degrees:  $V = V^0 \oplus V^+$ .

**Definition 2.1.** A *relative Sullivan model* (or *cofibration* in the category of cdgas) is a morphism of cdgas of the form

$$(B, d_B) \hookrightarrow (B \otimes \Lambda V, d), b \mapsto b \otimes 1$$

where

- $H^0(B) \cong \mathbf{k}$ ,
- $V = V^{\geq 1}$ ,
- and  $V$  is the direct sum of graded vector spaces  $V(k)$ :

$$\forall n, V^n = \bigoplus_{k \in \mathbb{N}} V(k)^n$$

such that  $d : V(0) \rightarrow B \otimes \mathbf{k}$  and  $d : V(k) \rightarrow B \otimes \Lambda(V(< k))$ . Here  $V(< k)$  denotes the direct sum  $V(0) \oplus \dots \oplus V(k-1)$ .

Let  $k \in \mathbb{N}$ . Denote by  $\Lambda^k V$  the sub graded vector space of  $\Lambda V$  generated by elements of the form  $v_1 \wedge \dots \wedge v_k$ ,  $v_i \in V$ . Elements of  $\Lambda^k V$  have by definition *wordlength*  $k$ . For example  $\Lambda V = \mathbf{k} \oplus V \oplus \Lambda^{\geq 2} V$ .

**Definition 2.2.** A relative Sullivan model  $(B, d_B) \hookrightarrow (B \otimes \Lambda V, d)$  is *minimal* if  $d : V \rightarrow B^+ \otimes \Lambda V + B \otimes \Lambda^{\geq 2} V$ . A (*minimal*) *Sullivan model* is a (minimal) relative Sullivan model of the form  $(B, d_B) = (\mathbf{k}, 0) \hookrightarrow (\Lambda V, d)$ .

**Example 2.3.** [5, end of the proof of Lemma 23.1] Let  $(\Lambda V, d)$  be cdga such that  $V = V^{\geq 2}$ . Then  $(\Lambda V, d)$  is a Sullivan model.

*proof assuming the minimality condition.* [5, p. 144] Suppose that  $d : V \rightarrow \Lambda^{\geq 2} V$ . In this case, the  $V(k)$  are easy to define: let  $V(k) := V^k$  for  $k \in \mathbb{N}$ . Let  $v \in V^k$ . By the minimality condition,  $dv$  is equal to a sum  $\sum_i x_i y_i$  where the non trivial elements  $x_i$  and  $y_i$  are both of positive length and therefore both of degree  $\geq 2$ . Since  $|x_i| + |y_i| = |dv| = k + 1$ , both  $x_i$  and  $y_i$  are of degree less than  $k$ . Therefore  $dv$  belongs to  $\Lambda(V^{<k}) = \Lambda(V(< k))$ .  $\square$

**Property 2.4.** The composite of relative Sullivan models is again a Sullivan relative model.

**Definition 2.5.** Let  $C$  be a cdga. A (minimal) *Sullivan model of  $C$*  is a (minimal) Sullivan model  $(\Lambda V, d)$  such that there exists a quasi-isomorphism of cdgas  $(\Lambda V, d) \xrightarrow{\sim} C$ .

Let  $\varphi : B \rightarrow C$  be a morphism of cdgas. A (minimal) *relative Sullivan model of  $\varphi$*  is a (minimal) relative Sullivan model  $(B, d_B) \hookrightarrow (B \otimes \Lambda V, d)$  such that  $\varphi$  can be decomposed as the composite of the relative Sullivan model and of a quasi-isomorphism of cdgas:

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & C \\ & \searrow & \uparrow \simeq \\ & & B \otimes \Lambda V \end{array}$$

**Theorem 2.6.** Any morphism  $\varphi : B \rightarrow C$  of cdgas admits a minimal relative Sullivan model if  $H^0(B) \cong \mathbf{k}$ ,  $H^0(\varphi)$  is an isomorphism and  $H^1(\varphi)$  is injective.

This theorem is proved in general by Proposition 14.3 and Theorem 14.9 of [5]. But in practice, if  $H^1(\varphi)$  is an isomorphism, we construct a minimal relative Sullivan model, by induction on degrees as in Proposition 12.2. of [5].

## 2.2 An example of relative Sullivan model

Consider the minimal Sullivan model of an odd sphere found in section 1.2

$$(\Lambda v, 0) \xrightarrow{\cong} A(S^{2n+1}).$$

Assume that  $n \geq 1$ . Consider the multiplication of  $\Lambda v$ : the morphism of cdgas

$$\mu : (\Lambda v_1, 0) \otimes (\Lambda v_2, 0) \rightarrow (\Lambda v, 0), v_1 \mapsto v, v_2 \mapsto v.$$

Recall that  $v, v_1$  and  $v_2$  are of degree  $2n + 1$ .

Denote by  $sv$  an element of degree  $|sv| = |s| + |v| = -1 + |v|$ . The operator  $s$  of degree  $-1$  is called the *suspension*.

We construct now a minimal relative Sullivan model of  $\mu$ . Define  $d(sv) = v_2 - v_1$ . Let  $m : \Lambda(v_1, v_2, sv), d \rightarrow (\Lambda v, 0)$  be the unique morphism of cdgas extending  $\mu$  such that  $m(sv) = 0$ .

$$\begin{array}{ccc} (\Lambda v_1, 0) \otimes (\Lambda v_2, 0) & \xrightarrow{\mu} & (\Lambda v, 0) \\ & \searrow & \uparrow m \\ & & \Lambda(v_1, v_2, sv, d) \end{array}$$

**Definition 2.7.** Let  $A$  be a differential graded algebra such that  $A^0 = \mathbf{k}$ . The complex of indecomposables of  $A$ , denoted  $Q(A)$ , is the quotient  $A^+/\mu(A^+ \otimes A^+)$ .

The complex of indecomposables of  $(\Lambda v, 0)$ ,  $Q((\Lambda v, 0))$ , is  $(\mathbf{k}v, 0)$  while

$$Q(\Lambda(v_1, v_2, sv, d)) = (\mathbf{k}v_1 \oplus \mathbf{k}v_2 \oplus \mathbf{k}sv, d(sv) = v_2 - v_1).$$

The morphism of complexes  $Q(m) : (\mathbf{k}v, 0) \rightarrow (\mathbf{k}v_1 \oplus \mathbf{k}v_2 \oplus \mathbf{k}sv, d(sv) = v_2 - v_1)$  map  $v_1$  to  $v$ ,  $v_2$  to  $v$  and  $sv$  to  $0$ . It is easy to check that  $Q(m)$  is a quasi-isomorphism of complexes.

By Proposition 14.13 of [5], since  $m$  is a morphism of cdgas between Sullivan model,  $Q(m)$  is a quasi-isomorphism if and only if  $m$  is a quasi-isomorphism.

So we have proved that  $m$  is a quasi-isomorphism and therefore

$$(\Lambda v_1, 0) \otimes (\Lambda v_2, 0) \hookrightarrow \Lambda(v_1, v_2, sv, d)$$

is a minimal relative Sullivan model of  $\mu$ . Consider the following commutative

diagram of cdgas where the square is a pushout

$$\begin{array}{ccc}
 & & \Lambda v, 0 \\
 & \nearrow \mu & \nearrow \simeq \\
 \Lambda(v_1, v_2), 0 & \longrightarrow & \Lambda(v_1, v_2, sv), d \\
 \downarrow \mu & & \downarrow \\
 \Lambda v, 0 & \longrightarrow & \Lambda v, 0 \otimes_{\Lambda(v_1, v_2), 0} \Lambda(v_1, v_2, sv), d
 \end{array}$$

It is easy to check that the cdga  $\Lambda v, 0 \otimes_{\Lambda(v_1, v_2), 0} \Lambda(v_1, v_2, sv), d$  is isomorphic to  $\Lambda(v, sv), 0$ . As we will explain later, we have computed in fact, the minimal Sullivan model  $\Lambda(v, sv), 0$  of the free loop space  $(S^{2n+1})^{S^1}$ . In particular, the cohomology algebra  $H^*((S^{2n+1})^{S^1}; \mathbf{k})$  is isomorphic to  $\Lambda(v, sv)$ . We can deduce easily that for  $p \in \mathbb{N}$ ,  $\dim H^p((S^{2n+1})^{S^1}) \leq 1$ . So we have shown that the sequence of Betti numbers of the free loop space on odd dimensional spheres is bounded.

### 2.3 The relative Sullivan model of the multiplication

**Proposition 2.8.** [6, Example 2.48] *Let  $(\Lambda V, d)$  be a relative minimal Sullivan model with  $V = V^{\geq 2}$  (concentrated in degrees  $\geq 2$ ). Then the multiplication  $\mu : (\Lambda V, d) \otimes (\Lambda V, d) \rightarrow (\Lambda V, d)$  admits a minimal relative Sullivan model of the form  $(\Lambda V \otimes \Lambda V \otimes \Lambda sV, D)$ .*

*Constructive proof.* We proceed by induction on  $n \in \mathbb{N}^*$  to construct quasi-isomorphisms of cdgas  $\varphi_n : (\Lambda V^{\leq n} \otimes \Lambda V^{\leq n} \otimes \Lambda sV^{\leq n}, D) \xrightarrow{\simeq} (\Lambda V^{\leq n}, d)$  extending the multiplication on  $\Lambda V^{\leq n}$ .

Suppose that  $\varphi_n$  is constructed. We now define  $\varphi_{n+1}$  extending  $\varphi_n$  and  $\mu$ , the multiplication on  $\Lambda V$ . Let  $v \in V^{n+1}$ . Then  $d(v) \in \Lambda^{\geq 2}(V^{\leq n})$  and  $\varphi_n(dv \otimes 1 \otimes 1 - 1 \otimes dv \otimes 1) = 0$ . Since  $\varphi_n$  is a surjective quasi-isomorphism, by the long exact sequence associated to a short exact sequence of complexes,  $\text{Ker } \varphi_n$  is acyclic. Therefore since  $dv \otimes 1 \otimes 1 - 1 \otimes dv \otimes 1$  is a cycle, there exists an element  $\gamma$  of degree  $n+1$  of  $\Lambda V^{\leq n} \otimes \Lambda V^{\leq n} \otimes \Lambda sV^{\leq n}$  such that  $D(\gamma) = dv \otimes 1 \otimes 1 - 1 \otimes dv \otimes 1$  and  $\varphi_n(\gamma) = 0$ . For degree reasons,  $\gamma$  is decomposable, i. e. has wordlength  $\geq 2$ . We define  $D(1 \otimes 1 \otimes sv) = v \otimes 1 \otimes 1 - 1 \otimes v \otimes 1 - \gamma$  and  $\varphi_{n+1}(1 \otimes 1 \otimes sv) = 0$ . Since  $D \circ D(1 \otimes 1 \otimes sv) = 0$  and  $d \circ \varphi_{n+1}(1 \otimes 1 \otimes sv) = \varphi_{n+1} \circ d(1 \otimes 1 \otimes sv)$ , by Property 1.6, the derivation  $D$  is a differential on  $\Lambda V^{\leq n+1} \otimes \Lambda V^{\leq n+1} \otimes \Lambda sV^{\leq n+1}$  and the morphism of graded algebras  $\varphi_{n+1}$  is a morphism of complexes.

The complex of indecomposables of  $(\Lambda V^{\leq n+1} \otimes \Lambda V^{\leq n+1} \otimes \Lambda sV^{\leq n+1}, D)$ ,

$$Q((\Lambda V^{\leq n+1} \otimes \Lambda V^{\leq n+1} \otimes \Lambda sV^{\leq n+1}, D))$$

is  $(V^{\leq n+1} \oplus V^{\leq n+1} \oplus sV^{\leq n+1}, d)$  with differential  $d$  given by  $d(v' \oplus v'' \oplus sv) = v \oplus -v \oplus 0$  for  $v', v''$  and  $v \in V^{\leq n+1}$ . Therefore it is easy to check that  $Q(\varphi_{n+1})$  is a quasi-isomorphism. So by Proposition 14.13 of [5],  $\varphi_{n+1}$  is a quasi-isomorphism. Since  $\gamma$  is of degree  $n+1$  and  $sV^{\leq n}$  is of degree  $< n$ , this relative Sullivan model



is minimal. We now define  $\varphi : (\Lambda V \otimes \Lambda V \otimes \Lambda sV, D) \rightarrow (\Lambda V, d)$  as

$$\lim_{\rightarrow} \varphi_n = \bigcup_{n \in \mathbb{N}} \varphi_n : \bigcup_{n \in \mathbb{N}} (\Lambda V^{\leq n} \otimes \Lambda V^{\leq n} \otimes \Lambda sV^{\leq n}) \rightarrow \bigcup_{n \in \mathbb{N}} \Lambda V^{\leq n}.$$

Since homology commutes with direct limits in the category of complexes [14, Chap 4, Sect 2, Theorem 7],  $H(\varphi) = \lim_{\rightarrow} H(\varphi_n)$  is an isomorphism.  $\square$

### 3 Rational homotopy theory

Let  $X$  be a topological space. Denote by  $S^*(X)$  the singular cochains of  $X$  with coefficients in  $\mathbf{k}$ . The dga  $S^*(X)$  is almost never commutative. Nevertheless, Sullivan, inspired by Quillen proved the following theorem.

**Theorem 3.1.** [5, Corollary 10.10] *For any topological space  $X$ , there exists two natural quasi-isomorphisms of dgas*

$$S^*(X) \xrightarrow{\sim} D(X) \xleftarrow{\sim} A_{PL}(X)$$

where  $A_{PL}(X)$  is commutative.

**Remark 3.2.** This cdga  $A_{PL}(X)$  is called the algebra of *polynomial differential forms*. If  $\mathbf{k} = \mathbb{R}$  and  $X$  is a smooth manifold  $M$ , you can think that  $A_{PL}(M)$  is the De Rham algebra of differential forms on  $M$ ,  $A_{DR}(M)$  [5, Theorem 11.4].

**Definition 3.3.** [6, Definition 2.34] Two topological spaces  $X$  and  $Y$  have the same *rational homotopy type* if there exists a finite sequence of continuous applications

$$X \xrightarrow{f_0} Y_1 \xleftarrow{f_1} Y_2 \dots Y_{n-1} \xrightarrow{f_{n-1}} Y_n \xrightarrow{f_n} Y$$

such that the induced maps in rational cohomology

$$H^*(X; \mathbb{Q}) \xleftarrow{H^*(f_0)} H^*(Y_1; \mathbb{Q}) \xrightarrow{H^*(f_1)} H^*(Y_2; \mathbb{Q}) \dots H^*(Y_{n-1}; \mathbb{Q}) \\ \xrightarrow{H^*(f_{n-1})} H^*(Y_n; \mathbb{Q}) \xleftarrow{H^*(f_n)} H^*(Y; \mathbb{Q})$$

are all isomorphisms.

**Theorem 3.4.** *Let  $X$  be a path connected topological space.*

1) (Unicity of minimal Sullivan models [5, Corollary p. 191]) *Two minimal Sullivan models of  $A_{PL}(X)$  are isomorphic.*

2) *Suppose that  $X$  is simply connected and  $\forall n \in \mathbb{N}$ ,  $H_n(X; \mathbf{k})$  is finite dimensional. Let  $(\Lambda V, d)$  be a minimal Sullivan model of  $X$ . Then [5, Theorem 15.11] for all  $n \in \mathbb{N}$ ,  $V^n$  is isomorphic to  $\text{Hom}_{\mathbf{k}}(\pi_n(X) \otimes_{\mathbb{Z}} \mathbf{k}, \mathbf{k}) \cong \text{Hom}_{\mathbb{Z}}(\pi_n(X), \mathbf{k})$ . In particular [5, Remark 1 p.208],  $\text{Dimension } V^n = \text{Dimension } \pi_n(X) \otimes_{\mathbb{Z}} \mathbf{k} < \infty$ .*

**Remark 3.5.** The isomorphism of graded vector spaces between  $V$  and  $\text{Hom}_{\mathbf{k}}(\pi_*(X) \otimes_{\mathbb{Z}} \mathbf{k}, \mathbf{k})$  is natural in some sense [6, p. 75-6] with respect to maps  $f : X \rightarrow Y$ . The

isomorphism behaves well also with respect to the long exact sequence associated to a (Serre) fibration ([5, Proposition 15.13] or [6, Proposition 2.65]).

**Theorem 3.6.** [6, Proposition 2.35][5, p. 139] *Let  $X$  and  $Y$  be two simply connected topological spaces such that  $H^n(X; \mathbb{Q})$  and  $H^n(Y; \mathbb{Q})$  are finite dimensional for all  $n \in \mathbb{N}$ . Let  $(\Lambda V, d)$  be a minimal Sullivan model of  $X$  and let  $(\Lambda W, d)$  be a minimal Sullivan model of  $Y$ . Then  $X$  and  $Y$  have the same rational homotopy type if and only if  $(\Lambda V, d)$  is isomorphic to  $(\Lambda W, d)$  as cdgas.*

## 4 Sullivan model of a pullback

### 4.1 Sullivan model of a product

Let  $X$  and  $Y$  be two topological spaces. Let  $p_1 : X \times Y \rightarrow Y$  and  $p_2 : X \times Y \rightarrow X$  be the projection maps. Let  $m$  be the unique morphism of cdgas given by the universal property of the tensor product (Example 1.4 1))

$$\begin{array}{ccc}
 & A_{PL}(Y) & \\
 & \downarrow & \searrow^{A_{PL}(p_2)} \\
 A_{PL}(X) & \longrightarrow & A_{PL}(X) \otimes A_{PL}(Y) \\
 & \searrow^{A_{PL}(p_1)} & \swarrow_{\exists! m} \\
 & & A_{PL}(X \times Y).
 \end{array}$$

Assume that  $H^*(X; \mathbf{k})$  or  $H^*(Y; \mathbf{k})$  is finite dimensional in all degrees. Then [5, Example 2, p. 142-3]  $m$  is a quasi-isomorphism. Let  $m_X : \Lambda V \xrightarrow{\cong} A_{PL}(X)$  be a Sullivan model of  $X$ . Let  $m_Y : \Lambda W \xrightarrow{\cong} A_{PL}(Y)$  be a Sullivan model of  $Y$ . Then by Künneth theorem, the composite

$$\Lambda V \otimes \Lambda W \xrightarrow{m_X \otimes m_Y} A_{PL}(X) \otimes A_{PL}(Y) \xrightarrow{m} A_{PL}(X \times Y)$$

is a quasi-isomorphism of cdgas. Therefore we have proved that “the Sullivan model of a product is the tensor product of the Sullivan models”.

### 4.2 the model of the diagonal

Let  $X$  be a topological space such that  $H^*(X)$  is finite dimensional in all degrees. Denote by  $\Delta : X \rightarrow X \times X$ ,  $x \mapsto (x, x)$  the diagonal map of  $X$ . Using the previous paragraph, since  $A_{PL}(p_1 \circ \Delta) = A_{PL}(p_2 \circ \Delta) = A_{PL}(\text{id}) = \text{id}$ , we have

the commutative diagram of cdgas.

$$\begin{array}{ccccc}
 A_{PL}(X) & \longrightarrow & A_{PL}(X) \otimes A_{PL}(X) & \longleftarrow & A_{PL}(X) \\
 & \searrow^{A_{PL}(p_1)} & \downarrow \simeq m & \swarrow_{A_{PL}(p_2)} & \\
 & & A_{PL}(X \times X) & & \\
 & \searrow^{\text{id}} & \downarrow A_{PL}(\Delta) & \swarrow_{\text{id}} & \\
 & & A_{PL}(X) & & 
 \end{array}$$

Therefore the composite  $A_{PL}(X) \otimes A_{PL}(X) \xrightarrow{m} A_{PL}(X \times X) \xrightarrow{A_{PL}(\Delta)} A_{PL}(X)$  coincides with the multiplication  $\mu : A_{PL}(X) \otimes A_{PL}(X) \rightarrow A_{PL}(X)$ . Therefore the following diagram of cdgas commutes

$$\begin{array}{ccc}
 A_{PL}(X) & \xleftarrow{A_{PL}(\Delta)} & A_{PL}(X \times X) \\
 \uparrow m_X & \swarrow \mu & \uparrow m \\
 A_{PL}(X) & & A_{PL}(X) \otimes A_{PL}(X) \\
 \uparrow \simeq & & \uparrow \simeq m_X \otimes m_X \\
 \Lambda V & \xleftarrow{\mu} & \Lambda V \otimes \Lambda V
 \end{array}$$

Here  $m_X : \Lambda V \xrightarrow{\simeq} A_{PL}(X)$  denotes a Sullivan model of  $X$ . Therefore we have proved that “the morphism modelling the diagonal map is the multiplication of the Sullivan model”.

### 4.3 Sullivan model of a fibre product

Consider a pullback square in the category of topological spaces

$$\begin{array}{ccc}
 P & \xrightarrow{g} & E \\
 q \downarrow & & \downarrow p \\
 X & \xrightarrow{f} & B
 \end{array}$$

where

- $p : E \rightarrow B$  is a (Serre) fibration between two topological spaces,
- for every  $i \in \mathbb{N}$ ,  $H^i(X)$  and  $H^i(B)$  are finite dimensional,
- the topological spaces  $X$  and  $E$  are path-connected and  $B$  is simply-connected.

Since  $p$  is a (Serre) fibration, the pullback map  $q$  is also a (Serre) fibration. Let  $A_{PL}(B) \otimes \Lambda V$  be a relative Sullivan model of  $A(p)$ . Consider the corresponding

commutative diagram of cdgas

$$\begin{array}{ccccc}
 & A_{PL}(B) & \xrightarrow{A_{PL}(f)} & A_{PL}(X) & \\
 & \downarrow & & \downarrow & \\
 A_{PL}(B) \otimes \Lambda V & \longrightarrow & A_{PL}(X) \otimes_{A_{PL}(B)} & A_{PL}(B) \otimes \Lambda V & \\
 \uparrow m \simeq & & & \uparrow \exists! m' & \\
 A_{PL}(E) & \xrightarrow{A_{PL}(g)} & & & A_{PL}(P)
 \end{array}$$

$A_{PL}(p)$  (curved arrow from  $A_{PL}(B)$  to  $A_{PL}(E)$ )  
 $A_{PL}(q)$  (curved arrow from  $A_{PL}(X)$  to  $A_{PL}(P)$ )

where the rectangle is a pushout and  $m'$  is given by the universal property. Explicitly, for  $x \in A_{PL}(X)$  and  $e \in A_{PL}(B) \otimes \Lambda V$ ,  $m'(x \otimes e)$  is the product of  $A_{PL}(q)(x)$  and  $A_{PL}(g) \circ m(e)$ .

Since  $A_{PL}(B) \hookrightarrow A_{PL}(B) \otimes \Lambda V$  is a relative Sullivan model, the inclusion obtained via pullback  $A_{PL}(X) \hookrightarrow A_{PL}(X) \otimes_{A_{PL}(B)} (A_{PL}(B) \otimes \Lambda V, d) \cong (A_{PL}(X) \otimes \Lambda V, d)$  is also a relative Sullivan model (minimal if  $A_{PL}(B) \hookrightarrow A_{PL}(B) \otimes \Lambda V$  is minimal).

By [5, Proposition 15.8] (or for weaker hypothesis [6, Theorem 2.70]),

**Theorem 4.1.** *The morphism of cdgas  $m'$  is a quasi-isomorphism.*

We can summarize this theorem by saying that: “The push-out of a (minimal) relative Sullivan model of a fibration is a (minimal) relative Sullivan model of the pullback of the fibration.”

*Idea of the proof.* Since by [5, Lemma 14.1],  $A_{PL}(B) \otimes \Lambda V$  is a “semi-free” resolution of  $A_{PL}(E)$  as left  $A_{PL}(B)$ -modules, by definition of the differential torsion product,

$$\mathrm{Tor}^{A_{PL}(B)}(A_{PL}(X), A_{PL}(E)) := H(A_{PL}(X) \otimes_{A_{PL}(B)} (A_{PL}(B) \otimes \Lambda V)).$$

By Theorem 3.1 and naturality, we have an isomorphism of graded vector spaces

$$\mathrm{Tor}^{A_{PL}(B)}(A_{PL}(X), A_{PL}(E)) \cong \mathrm{Tor}^{S^*(B)}(S^*(X), S^*(E)).$$

The Eilenberg-Moore formula gives an isomorphism of graded vector spaces

$$\mathrm{Tor}^{S^*(B)}(S^*(X), S^*(E)) \cong H^*(P).$$

We claimed that the resulting isomorphism between the homology of  $A_{PL}(X) \otimes_{A_{PL}(B)} (A_{PL}(B) \otimes \Lambda V)$  and  $H^*(P)$  can be identified with  $H(m)$ . Therefore  $m$  is a quasi-isomorphism.  $\square$

Instead of working with  $A_{PL}$ , we prefer usually to work at the level of Sullivan models. Let  $m_B : \Lambda B \xrightarrow{\simeq} A_{PL}(B)$  be a Sullivan model of  $B$ . Let  $m_X : \Lambda X \xrightarrow{\simeq} A_{PL}(X)$  be a Sullivan model of  $X$ . Let  $\varphi$  be a morphism of cdgas such the following diagram commutes exactly

$$\begin{array}{ccc}
A_{PL}(B) & \xrightarrow{A_{PL}(f)} & A_{PL}(X) \\
m_B \uparrow \simeq & & m_X \uparrow \simeq \\
\Lambda B & \xrightarrow{\varphi} & \Lambda X
\end{array}$$

Let  $\Lambda B \hookrightarrow \Lambda B \otimes \Lambda V$  be a relative Sullivan model of  $A_{PL}(p) \circ m_B$ . Consider the corresponding commutative diagram of cdgas

$$\begin{array}{ccccccc}
A_{PL}(B) & \xleftarrow[m_B]{\simeq} & \Lambda B & \xrightarrow{\varphi} & \Lambda X & \xrightarrow[m_X]{\simeq} & A_{PL}(X) & (1) \\
\downarrow A_{PL}(p) & & \downarrow & & \downarrow & & \downarrow A_{PL}(q) & \\
& & \Lambda B \otimes \Lambda V & \longrightarrow & \Lambda X \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V) & & & \\
& \swarrow m & & & \searrow \exists! m' & & & \\
A_{PL}(E) & \xrightarrow[\simeq]{m} & & \xrightarrow{A_{PL}(g)} & & & A_{PL}(P) & 
\end{array}$$

where the rectangle is a pushout and  $m'$  is given by the universal property. Then again,  $\Lambda X \hookrightarrow \Lambda X \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V)$  is a relative Sullivan model and the morphism of cdgas  $m'$  is a quasi-isomorphism.

The reader should skip the following remark on his first reading.

**Remark 4.2.** 1) In the previous proof, if the composites  $m_X \circ \varphi$  and  $A_{PL}(f) \circ m_B$  are not strictly equal then the map  $m'$  is not well defined. In general, the composites  $m_X \circ \varphi$  and  $A_{PL}(f) \circ m_B$  are only homotopic and the situation is more complicated: see part 2) of this remark.

2) Let  $m_B : \Lambda B \xrightarrow{\simeq} A_{PL}(B)$  be a Sullivan model of  $B$ . Let  $m'_X : \Lambda X' \xrightarrow{\simeq} A_{PL}(X)$  be a Sullivan model of  $X$ . By the lifting Lemma of Sullivan models [5, Proposition 14.6], there exists a morphism of cdgas  $\varphi' : \Lambda B \rightarrow \Lambda X'$  such that the following diagram commutes only up to homotopy (in the sense of [6, Section 2.2])

$$\begin{array}{ccc}
A_{PL}(B) & \xrightarrow{A_{PL}(f)} & A_{PL}(X) \\
m_B \uparrow \simeq & & m'_X \uparrow \simeq \\
\Lambda B & \xrightarrow{\varphi'} & \Lambda X'
\end{array}$$

In general, this square is not strictly commutative. Let  $\Lambda B \hookrightarrow \Lambda B \otimes \Lambda V$  be a relative Sullivan model of  $A_{PL}(p) \circ m_B$ . Then there exists a commutative diagram

of cdgas

$$\begin{array}{ccc}
A_{PL}(X) & \xrightarrow{A_{PL}(q)} & A_{PL}(P) \\
\uparrow \simeq & & \uparrow \simeq \\
\Lambda X & \longrightarrow & \Lambda X \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V) \\
\downarrow \simeq & & \downarrow \simeq \\
\Lambda X' & \longrightarrow & \Lambda X' \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V)
\end{array}$$

*Proof of part 2) of Remark 4.2.* Let  $\Lambda B \xrightarrow{\varphi} \Lambda X \xrightarrow{\theta} \Lambda X'$  be a relative Sullivan model of  $\varphi'$ . Since the composites  $m'_X \circ \theta \circ \varphi$  and  $A_{PL}(f) \circ m_B$  are homotopic, by the homotopy extension property [6, Proposition 2.22] of the relative Sullivan model  $\varphi : \Lambda B \hookrightarrow \Lambda X$ , there exists a morphism of cdgas  $m_X : \Lambda X \rightarrow A_{PL}(X)$  homotopic to  $m'_X \circ \theta$  such that  $m_X \circ \varphi = A_{PL}(f) \circ m_B$ . Therefore using diagram (1), we obtain the following commutative diagram of cdgas:

$$\begin{array}{ccccc}
A_{PL}(X) & \xrightarrow{A_{PL}(q)} & A_{PL}(P) & \xleftarrow{A_{PL}(g)} & A_{PL}(E) \\
\uparrow \simeq m_X & & \uparrow \simeq m' & & \uparrow \simeq m \\
\Lambda X & \longrightarrow & \Lambda X \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V) & \longleftarrow & \Lambda B \otimes \Lambda V \\
\downarrow \simeq \theta & & \downarrow \simeq \theta \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V) & & \\
\Lambda X' & \longrightarrow & \Lambda X' \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V) & & 
\end{array}$$

Here, since  $\theta$  is a quasi-isomorphism, the pushout morphism  $\theta \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V)$  along the relative Sullivan model  $\Lambda X \hookrightarrow \Lambda X \otimes_{\Lambda B} (\Lambda B \otimes \Lambda V)$  is also a quasi-isomorphism [5, Lemma 14.2].  $\square$

#### 4.4 Sullivan model of a fibration

Let  $p : E \rightarrow B$  be a (Serre) fibration with fibre  $F := p^{-1}(b_0)$ .

$$\begin{array}{ccc}
F & \xrightarrow{j} & E \\
\downarrow & & \downarrow p \\
b_0 & \longrightarrow & B
\end{array}$$

Taking  $X$  to be the point  $b_0$ , we can apply the results of the previous section. Let  $m_B : (\Lambda V, d) \xrightarrow{\simeq} A_{PL}(B)$  be a Sullivan model of  $B$ . Let  $(\Lambda V, d) \hookrightarrow (\Lambda V \otimes \Lambda W, d)$  be a relative Sullivan model of  $A_{PL}(p) \circ m_B$ .

Since  $A_{PL}(\{b_0\})$  is equal to  $(\mathbf{k}, 0)$ , there is a unique morphism of cdgas  $m'$  such that the following diagram commutes

$$\begin{array}{ccccc}
A_{PL}(B) & \xrightarrow{A_{PL}(p)} & A_{PL}(E) & \xrightarrow{A_{PL}(j)} & A_{PL}(F) \\
\uparrow \simeq m_B & & \uparrow \simeq & & \uparrow m' \\
(\Lambda V, d) & \longrightarrow & (\Lambda V \otimes \Lambda W, d) & \longrightarrow & (k, 0) \otimes_{(\Lambda V, d)} (\Lambda V \otimes \Lambda W, d)
\end{array}$$

Suppose that the base  $B$  is a simply connected space and that the total space  $E$  is path-connected. Then by the previous section, the morphism of cdga's

$$m' : (k, 0) \otimes_{(\Lambda V, d)} (\Lambda V \otimes \Lambda W, d) \cong (\Lambda W, \bar{d}) \xrightarrow{\simeq} A_{PL}(F)$$

is a quasi-isomorphism:

“The cofiber of a relative Sullivan model of a fibration is a Sullivan model of the fiber of the fibration.”

Note that the cofiber of a relative Sullivan model is minimal if and only if the relative Sullivan model is minimal.

## 4.5 Sullivan model of free loop spaces

Let  $X$  be a simply-connected space. Consider the commutative diagram of spaces

$$\begin{array}{ccccc}
X^{S^1} & \longrightarrow & X^I & \xleftarrow[\simeq]{\sigma} & X \\
\downarrow ev & & \downarrow (ev_0, ev_1) & \swarrow \Delta & \\
X & \xrightarrow[\Delta]{} & X \times X & & 
\end{array}$$

where the square is a pullback. Here  $I$  denotes the closed interval  $[0, 1]$ ,  $ev$ ,  $ev_0$ ,  $ev_1$  are the evaluation maps and the homotopy equivalence  $\sigma : X \xrightarrow{\simeq} X^I$  is the inclusion of constant paths. Let  $m_X : \Lambda V \xrightarrow{\simeq} A_{PL}(X)$  be a minimal Sullivan model of  $X$ . By Proposition 2.8, the multiplication  $\mu : \Lambda V \otimes \Lambda V \rightarrow \Lambda V$  admits a minimal relative Sullivan model of the form

$$\Lambda V \otimes \Lambda V \hookrightarrow \Lambda V \otimes \Lambda V \otimes \Lambda sV.$$

Since  $\mu$  is a model of the diagonal (Section 4.2) and since  $\Delta = (ev_0, ev_1) \circ \sigma$ , we have the commutative rectangle of cdgas

$$\begin{array}{ccccc}
A_{PL}(X \times X) & \xrightarrow{A_{PL}((ev_0, ev_1))} & A_{PL}(X^I) & \xrightarrow{A_{PL}(\sigma)} & A_{PL}(X) \\
\uparrow m_{X \times X} \simeq & & & & \uparrow m_X \simeq \\
\Lambda V \otimes \Lambda V & \longrightarrow & \Lambda V \otimes \Lambda V \otimes \Lambda sV & \xrightarrow{\simeq} & \Lambda V
\end{array}$$

Since  $\sigma$  is a homotopy equivalence,  $S^*(\sigma)$  is a homotopy equivalence of complexes and in particular a quasi-isomorphism. So by Theorem 3.1 and naturality,  $A_{PL}(\sigma)$  is also a quasi-isomorphism. Therefore, by the lifting property of relative Sullivan models [5, Proposition 14.6], there exists a morphism of cdgas  $\varphi : \Lambda V \otimes \Lambda V \otimes \Lambda sV \rightarrow \Lambda V$

$\Lambda sV \rightarrow A_{PL}(X^I)$  such that, in the diagram of cdgas

$$\begin{array}{ccccc} A_{PL}(X \times X) & \xrightarrow{A_{PL}((ev_0, ev_1))} & A_{PL}(X^I) & \xrightarrow[A_{PL}(\sigma)]{\simeq} & A_{PL}(X) \\ m_{X \times X} \uparrow \simeq & & \varphi \uparrow \simeq & & \simeq \uparrow m_X \\ \Lambda V \otimes \Lambda V & \longrightarrow & \Lambda V \otimes \Lambda V \otimes \Lambda sV & \xrightarrow[\simeq]{} & \Lambda V \end{array}$$

the left square commutes exactly and the right square commutes in homology. Therefore  $\varphi$  is also a quasi-isomorphism. This means that

$$\Lambda V \otimes \Lambda V \hookrightarrow \Lambda V \otimes \Lambda V \otimes \Lambda sV.$$

is a relative Sullivan model of the composite

$$\Lambda V \otimes \Lambda V \xrightarrow{m_{X \times X}} A_{PL}(X \times X) \xrightarrow{A_{PL}((ev_0, ev_1))} A_{PL}(X^I).$$

Here diagram (1) specializes to the following commutative diagram of cdgas

$$\begin{array}{ccccc} \Lambda V \otimes \Lambda V & \xrightarrow{\mu} & \Lambda V & \xrightarrow[\simeq]{m_X} & A_{PL}(X) \\ \downarrow & & \downarrow & & \downarrow A_{PL}(ev) \\ \Lambda V \otimes \Lambda V \otimes \Lambda sV & \longrightarrow & \Lambda V \otimes_{\Lambda V \otimes \Lambda V} \Lambda V \otimes \Lambda V \otimes \Lambda sV & \xrightarrow[\simeq]{} & A(X^{S^1}) \\ \swarrow \varphi \simeq & \xrightarrow{\quad} & \searrow \simeq & & \downarrow \\ A(X^I) & \xrightarrow{\quad} & & & A(X^{S^1}) \end{array} \quad (2)$$

where the rectangle is a pushout. Therefore

$$\Lambda V \hookrightarrow \Lambda V \otimes_{\Lambda V \otimes \Lambda V} (\Lambda V \otimes \Lambda V \otimes \Lambda sV) \cong (\Lambda V \otimes \Lambda sV, \delta)$$

is a minimal relative Sullivan model of  $A_{PL}(ev) \circ m_X$ .

**Corollary 4.3.** *Let  $X$  be a simply-connected space. Then the free loop space cohomology of  $H^*(X^{S^1}; \mathbf{k})$  with coefficients in a field  $\mathbf{k}$  of characteristic 0 is isomorphic to the Hochschild homology of  $A_{PL}(X)$ ,  $HH_*(A_{PL}(X), A_{PL}(X))$ .*

Replacing  $A_{PL}(X)$  by  $A_{DR}(M)$  (Remark 3.2), this Corollary is a theorem of Chen [3, 3.2.3 Theorem] when  $X$  is a smooth manifold  $M$ .

*Proof.* The quasi-isomorphism of cdgas  $m_X : \Lambda V \xrightarrow{\simeq} A_{PL}(X)$  induces an isomorphism between Hochschild homologies

$$HH_*(m_X, m_X) : HH_*(\Lambda V, \Lambda V) \xrightarrow{\cong} HH_*(A_{PL}(X), A_{PL}(X)).$$

By [5, Lemma 14.1],  $\Lambda V \otimes \Lambda V \otimes \Lambda sV$  is a semi-free resolution of  $\Lambda V$  as a  $\Lambda V \otimes \Lambda V^{op}$ -module. Therefore the Hochschild homology  $HH_*(\Lambda V, \Lambda V)$  can be defined as the homology of the cdga  $(\Lambda V \otimes \Lambda sV, \delta)$ . We have just seen above that  $H(\Lambda V \otimes \Lambda sV, \delta)$  is isomorphic to the free loop space cohomology  $H^*(X^{S^1}; \mathbf{k})$ .  $\square$



We have shown that a Sullivan model of  $X^{S^1}$  is of the form  $(\Lambda V \otimes \Lambda sV, \delta)$ . The following theorem of Vigué-Poirrier and Sullivan gives a precise description of the differential  $\delta$ .

**Theorem 4.4.** (*[17, Theorem p. 637] or [6, Theorem 5.11]*) *Let  $X$  be a simply connected topological space. Let  $(\Lambda V, d)$  be a minimal Sullivan model of  $X$ . For all  $v \in V$ , denote by  $sv$  an element of degree  $|v| - 1$ . Let  $s : \Lambda V \otimes \Lambda sV \rightarrow \Lambda V \otimes \Lambda sV$  be the unique derivation of (upper) degree  $-1$  such that on the generators  $v, sv, v \in V, s(v) = sv$  and  $s(sv) = 0$ . We have  $s \circ s = 0$ . Then there exists a unique Sullivan model of  $X^{S^1}$  of the form  $(\Lambda V \otimes \Lambda sV, \delta)$  such that  $\delta \circ s + s \circ \delta = 0$  on  $\Lambda V \otimes \Lambda sV$ .*

**Remark 4.5.** Consider the free loop fibration  $\Omega X \hookrightarrow X^{S^1} \xrightarrow{ev} X$ . Since  $(\Lambda V, d) \hookrightarrow (\Lambda V \otimes \Lambda sV, \delta)$  is a minimal relative Sullivan model of  $A_{PL}(ev) \circ m_X$ , by Section 4.4,

$$\mathbb{k} \otimes_{(\Lambda V, d)} (\Lambda V \otimes \Lambda sV, \delta) \cong (\Lambda sV, \bar{\delta})$$

is a minimal Sullivan model of  $\Omega X$ . Let  $v \in V$ . By Theorem 4.4,  $\delta(sv) = -s\delta v = -sdv$ . Since  $dv \in \Lambda^{\geq 2}V$ ,  $\delta(sv) \in \Lambda^{\geq 1}V \otimes \Lambda^1 sV$ . Therefore  $\bar{\delta} = 0$ . Since  $\Omega X$  is a  $H$ -space, this follows also from Theorem 5.3 and from the unicity of minimal Sullivan models (part 1) of Theorem 3.4).

## 5 Examples of Sullivan models

### 5.1 Sullivan model of spaces with polynomial cohomology

The following proposition is a straightforward generalisation [5, p. 144] of the Sullivan model of odd-dimensional spheres (see section 1.2).

**Proposition 5.1.** *Let  $X$  be a path connected topological space such that its cohomology  $H^*(X; \mathbf{k})$  is a free graded commutative algebra  $\Lambda V$  (for example, polynomial). Then a Sullivan model of  $X$  is  $(\Lambda V, 0)$ .*

**Example 5.2.** Odd-dimensional spheres  $S^{2n+1}$ , complex or quaternionic Stiefel manifolds [6, Example 2.40]  $V_k(\mathbb{C}^n)$  or  $V_k(\mathbb{H}^n)$ , classifying spaces  $BG$  of simply connected Lie groups [6, Example 2.42], connected Lie groups  $G$  as we will see in the following section.

### 5.2 Sullivan model of an $H$ -space

An  $H$ -space is a pointed topological space  $(G, e)$  equipped with a pointed continuous map  $\mu : (G, e) \times (G, e) \rightarrow (G, e)$  such that the two pointed maps  $g \mapsto \mu(e, g)$  and  $g \mapsto \mu(g, e)$  are pointed homotopic to the identity map of  $(G, e)$ .

**Theorem 5.3.** [5, Example 3 p. 143] *Let  $G$  be a path connected  $H$ -space such that  $\forall n \in \mathbb{N}$ ,  $H_n(G; \mathbf{k})$  is finite dimensional. Then*

- 1) *its cohomology  $H^*(G; \mathbf{k})$  is a free graded commutative algebra  $\Lambda V$ ,*
- 2)  *$G$  has a Sullivan model of the form  $(\Lambda V, 0)$ , that is with zero differential.*

*Proof.* 1) Let  $A$  be  $H^*(G; \mathbf{k})$  the cohomology of  $G$ . By hypothesis,  $A$  is a connected commutative graded Hopf algebra (not necessarily associative). Now the theorem of Hopf-Borel in characteristic 0 [4, VII.10.16] says that  $A$  is a free graded commutative algebra.

2) By Proposition 5.1, 1) and 2) are equivalent.  $\square$

**Example 5.4.** Let  $G$  be a path-connected Lie group (or more generally a  $H$ -space with finitely generated integral homology). Then  $G$  has a Sullivan model of the form  $(\Lambda V, 0)$ . By Theorem 3.4,  $V^n$  and  $\pi_n(G) \otimes_{\mathbb{Z}} \mathbf{k}$  have the same dimension for any  $n \in \mathbb{N}$ . Since  $H_*(G; \mathbf{k})$  is of finite (total) dimension,  $V$  and therefore  $\pi_*(G) \otimes_{\mathbb{Z}} \mathbf{k}$  are concentrated in odd degrees. In fact, more generally [2, Theorem 6.11],  $\pi_2(G) = \{0\}$ . Note, however that  $\pi_4(S^3) = \mathbb{Z}/2\mathbb{Z} \neq \{0\}$ .

### 5.3 Sullivan model of projective spaces

Consider the complex projective space  $\mathbb{C}\mathbb{P}^n$ ,  $n \geq 1$ . The construction of the Sullivan model of  $\mathbb{C}\mathbb{P}^n$  is similar to the construction of the Sullivan model of  $S^2 = \mathbb{C}\mathbb{P}^1$  done in section 1.2:

The cohomology algebra  $H^*(A_{PL}(\mathbb{C}\mathbb{P}^n)) \cong H^*(\mathbb{C}\mathbb{P}^n)$  is the truncated polynomial algebra  $\frac{\mathbf{k}[x]}{x^{n+1}=0}$  where  $x$  is an element of degree 2. Let  $v$  be a cycle of  $A_{PL}(\mathbb{C}\mathbb{P}^n)$  representing  $x := [v]$ . The inclusion of complexes  $(\mathbf{k}v, 0) \hookrightarrow A_{PL}(\mathbb{C}\mathbb{P}^n)$  extends to a unique morphism of cdgas  $m : (\Lambda v, 0) \rightarrow A_{PL}(\mathbb{C}\mathbb{P}^n)$  (Property 1.6). Since  $[v^{n+1}] = x^{n+1} = 0$ , there exists an element  $\psi \in A_{PL}(\mathbb{C}\mathbb{P}^n)$  of degree  $2n+1$  such that  $d\psi = v^{n+1}$ . Let  $w$  denote another element of degree  $2n+1$ . Let  $d$  be the unique derivation of  $\Lambda(v, w)$  such that  $d(v) = 0$  and  $d(w) = v^{n+1}$ . The unique morphism of graded algebras  $m : (\Lambda(v, w), d) \rightarrow A_{PL}(\mathbb{C}\mathbb{P}^n)$  such that  $m(v) = v$  and  $m(w) = \psi$ , is a morphism of cdgas. In homology,  $H(m)$  sends  $1, [v], \dots, [v^n]$  to  $1, x, \dots, x^n$ . Therefore  $m$  is a quasi-isomorphism.

More generally, let  $X$  be a simply connected space such that  $H^*(X)$  is a truncated polynomial algebra  $\frac{\mathbf{k}[x]}{x^{n+1}=0}$  where  $n \geq 1$  and  $x$  is an element of even degree  $d \geq 2$ . Then the Sullivan model of  $X$  is  $(\Lambda(v, w), d)$  where  $v$  is an element of degree  $d$ ,  $w$  is an element of degree  $d(n+1) - 1$ ,  $d(v) = 0$  and  $d(w) = v^{n+1}$ .

### 5.4 Free loop space cohomology for even-dimensional spheres and projective spaces

In this section, we compute the free loop space cohomology of any simply connected space  $X$  whose cohomology is a truncated polynomial algebra  $\frac{\mathbf{k}[x]}{x^{n+1}=0}$  where  $n \geq 1$  and  $x$  is an element of even degree  $d \geq 2$ .

Mainly, this is the even-dimensional sphere  $S^d$  ( $n = 1$ ), the complex projective space  $\mathbb{C}\mathbb{P}^n$  ( $d = 2$ ), the quaternionic projective space  $\mathbb{H}\mathbb{P}^n$  ( $d = 4$ ) and the Cayley plane  $\mathbb{O}\mathbb{P}^2$  ( $n = 2$  and  $d = 8$ ).

In the previous section, we have seen that the minimal Sullivan model of  $X$  is  $(\Lambda(v, w), d(v) = 0, d(w) = v^{n+1})$  where  $v$  is an element of degree  $d$  and  $w$  is an element of degree  $d(n+1) - 1$ . By the constructive proof of Proposition 2.8, the multiplication  $\mu$  of this minimal Sullivan model  $(\Lambda(v, w), d)$  admits the relative Sullivan model  $(\Lambda(v, w) \otimes \Lambda(v, w) \otimes \Lambda(sv, sw), D)$  where

$$D(1 \otimes 1 \otimes sv) = v \otimes 1 \otimes 1 - 1 \otimes v \otimes 1 \text{ and}$$

$$D(1 \otimes 1 \otimes sw) = w \otimes 1 \otimes 1 - 1 \otimes w \otimes 1 - \sum_{i=0}^n v^i \otimes v^{n-i} \otimes sv.$$

Therefore, by taking the pushout along  $\mu$  of this relative Sullivan model (diagram (2)), or simply by applying Theorem 4.4, a relative Sullivan model of  $A_{PL}(ev) \circ m_X$  is given by the inclusion of cdgas  $(\Lambda(v, w), d) \hookrightarrow (\Lambda(v, w, sv, sw), \delta)$  where  $\delta(sv) = -sd(v) = 0$  and  $\delta(sw) = -s(v^{n+1}) = -(n+1)v^n sv$ . Consider the pushout square of cdgas

$$\begin{array}{ccc} (\Lambda(v, w), d) & \longrightarrow & (\Lambda(v, w, sv, sw), \delta) \\ \simeq \downarrow \theta & & \simeq \downarrow \theta \otimes_{\Lambda(v, w)} \Lambda(sv, sw) \\ \left( \frac{\mathbf{k}[v]}{v^{n+1}=0}, 0 \right) & \longrightarrow & \left( \frac{\mathbf{k}[v]}{v^{n+1}=0} \otimes \Lambda(sv, sw), \bar{\delta} \right). \end{array}$$

Here, since  $\theta$  is a quasi-isomorphism, the pushout morphism  $\theta \otimes_{\Lambda(v, w)} \Lambda(sv, sw)$  along the relative Sullivan model  $\Lambda(v, w) \hookrightarrow \Lambda(v, w, sv, sw)$  is also a quasi-isomorphism [5, Lemma 14.2]. Therefore,  $H^*(X^{S^1}; \mathbf{k})$  is the graded vector space

$$\mathbf{k} \oplus \bigoplus_{1 \leq p \leq n, i \in \mathbb{N}} \mathbf{k}v^p(sv)^i \oplus \bigoplus_{0 \leq p \leq n-1, i \in \mathbb{N}} \mathbf{k}v^p sv(sv)^i.$$

(In [11, Section 8], the author extends these rational computations over any commutative ring.) Since for all  $i \in \mathbb{N}$ , the degree of  $v(sv)^{i+1}$  is strictly greater than the degree of  $v^n(sv)^i$ , the generators  $1, v^p(sv)^i, 1 \leq p \leq n, i \in \mathbb{N}$ , have all distinct (even) degrees. Since for all  $i \in \mathbb{N}$ , the degree of  $sv(sv)^{i+1}$  is strictly greater than the degree of  $v^{n-1}sv(sv)^i$ , the generators  $v^p sv(sv)^i, 0 \leq p \leq n-1, i \in \mathbb{N}$ , have also distinct (odd) degrees. Therefore, for all  $p \in \mathbb{N}$ ,  $\text{Dim } H^p(X^{S^1}; \mathbf{k}) \leq 1$ .

At the end of section 2.2, we have shown the same inequalities when  $X$  is an odd-dimensional sphere, or more generally for a simply-connected space  $X$  whose cohomology  $H^*(X; \mathbf{k})$  is an exterior algebra  $\Lambda x$  on an odd degree generator  $x$ . Since every finite dimensional graded commutative algebra generated by a single element  $x$  is either  $\Lambda x$  or  $\frac{\mathbf{k}[x]}{x^{n+1}=0}$ , we have shown the following proposition:

**Proposition 5.5.** *Let  $X$  be a simply connected topological space such that its cohomology  $H^*(X; \mathbf{k})$  is generated by a single element and is finite dimensional. Then the sequence of Betti numbers of the free loop space on  $X$ ,  $b_n := \dim H^n(X^{S^1}; \mathbf{k})$*

is bounded.

The goal of the following section will be to prove the converse of this proposition.

## 6 Vigué-Poirrier-Sullivan theorem on closed geodesics

The goal of this section is to prove (See section 6.4) the following theorem due to Vigué-Poirrier and Sullivan.

### 6.1 Statement of Vigué-Poirrier-Sullivan theorem and of its generalisations

**Theorem 6.1.** ([17, Theorem p. 637] or [6, Proposition 5.14]) *Let  $M$  be a simply connected topological space such that the rational cohomology of  $M$ ,  $H^*(M; \mathbb{Q})$  is of finite (total) dimension (in particular, vanishes in higher degrees).*

*If the cohomology algebra  $H^*(M; \mathbb{Q})$  requires at least two generators then the sequence of Betti numbers of the free loop space on  $M$ ,  $b_n := \dim H^n(M^{S^1}; \mathbb{Q})$  is unbounded.*

**Example 6.2.** (Betti numbers of  $(S^3 \times S^3)^{S^1}$  over  $\mathbb{Q}$ )

Let  $V$  and  $W$  be two graded vector spaces such  $\forall n \in \mathbb{N}$ ,  $V^n$  and  $W^n$  are finite dimensional. We denote by

$$P_V(z) := \sum_{n=0}^{+\infty} (\dim V^n) z^n$$

the sum of the *Poincaré serie* of  $V$ . If  $V$  is the cohomology of a space  $X$ , we denote  $P_{H^*(X)}(z)$  simply by  $P_X(z)$ . Note that  $P_{V \otimes W}(z)$  is the product  $P_V(z)P_W(z)$ . We saw at the end of section 2.2 that  $H^*((S^3)^{S^1}; \mathbb{Q}) \cong \Lambda v \otimes \Lambda s v$  where  $v$  is an element of degree 3. Therefore

$$P_{(S^3)^{S^1}}(z) = (1 + z^3) \sum_{n=0}^{+\infty} z^{2n} = \frac{1 + z^3}{1 - z^2}.$$

Since the free loops on a product is the product of the free loops

$$H^*((S^3 \times S^3)^{S^1}) \cong H^*((S^3)^{S^1}) \otimes H^*((S^3)^{S^1}).$$

Therefore, since  $\frac{1}{1-z^2} = \sum_{n=0}^{+\infty} (n+1)z^{2n}$ ,

$$P_{(S^3 \times S^3)^{S^1}}(z) = \left( \frac{1+z^3}{1-z^2} \right)^2 = 1 + 2z^2 + \sum_{n=3}^{+\infty} (n-1)z^n.$$

So the Betti numbers over  $\mathbb{Q}$  of the free loop space on  $S^3 \times S^3$ ,  $b_n := \text{Dim } H^n((S^3 \times S^3)^{S^1}; \mathbb{Q})$  are equal to  $n-1$  if  $n \geq 3$ . In particular, they are unbounded.

**Conjecture 6.3.** *The theorem of Vigué-Poirrier and Sullivan holds replacing  $\mathbb{Q}$  by any field  $\mathbb{F}$ .*

**Example 6.4.** (Betti numbers of  $(S^3 \times S^3)^{S^1}$  over  $\mathbb{F}$ )

The calculation of Example 6.2 over  $\mathbb{Q}$  can be extended over any field  $\mathbb{F}$  as follows: Since  $S^3$  is a topological group, the map  $\Omega S^3 \times S^3 \rightarrow (S^3)^{S^1}$ , sending  $(w, g)$  to the free loop  $t \mapsto w(t)g$ , is a homeomorphism. Using Serre spectral sequence ([13, Proposition 17] or [14, Chap 9. Sect 7. Lemma 3]) or Bott-Samelson theorem ([12, Corollary 7.3.3] or [9, Appendix 2 Theorem 1.4]), the cohomology of the pointed loops on  $S^3$ ,  $H^*(\Omega S^3)$  is again isomorphic (as graded vector spaces only!) to the polynomial algebra  $\Lambda sv$  where  $sv$  is of degree 2. Therefore exactly as over  $\mathbb{Q}$ ,  $H^*((S^3)^{S^1}; \mathbb{F}) \cong \Lambda v \otimes \Lambda sv$  where  $v$  is an element of degree 3. Now the same proof as in Example 6.2 shows that the Betti numbers over  $\mathbb{F}$  of the free loop space on  $S^3 \times S^3$ ,  $b_n := \text{Dim } H^n((S^3 \times S^3)^{S^1}; \mathbb{F})$  are again equal to  $n-1$  if  $n \geq 3$ .

In fact, the theorem of Vigué-Poirrier and Sullivan is completely algebraic:

**Theorem 6.5.** ([17] when  $\mathbb{F} = \mathbb{Q}$ , [7, Theorem III p. 315] over any field  $\mathbb{F}$ ) *Let  $\mathbb{F}$  be a field. Let  $A$  be a cdga such that  $H^{<0}(A) = 0$ ,  $H^0(A) = \mathbb{F}$  and  $H^*(A)$  is of finite (total) dimension. If the algebra  $H^*(A)$  requires at least two generators then the sequence of dimensions of the Hochschild homology of  $A$ ,  $b_n := \dim HH_{-n}(A, A)$  is unbounded.*

Generalising Chen's theorem (Corollary 4.3) over any field  $\mathbb{F}$ , Jones theorem [10] gives the isomorphisms of vector spaces

$$H^n(X^{S^1}; \mathbb{F}) \cong HH_{-n}(S^*(X; \mathbb{F}), S^*(X; \mathbb{F})), \quad n \in \mathbb{Z}$$

between the free loop space cohomology of  $X$  and the Hochschild homology of the algebra of singular cochains on  $X$ . But since the algebra of singular cochains  $S^*(X; \mathbb{F})$  is not commutative, Conjecture 6.3 does not follow from Theorem 6.5.

## 6.2 A first result of Sullivan

In this section, we start by a first result of Sullivan whose simple proof illustrates the technics used in the proof of Vigué-Poirrier-Sullivan theorem.

**Theorem 6.6.** [15] *Let  $X$  be a simply-connected space such that  $H^*(X; \mathbb{Q})$  is not concentrated in degree 0 and  $H^n(X; \mathbb{Q})$  is null for  $n$  large enough. Then on the contrary,  $H^n(X^{S^1}; \mathbb{Q}) \neq 0$  for an infinite set of integers  $n$ .*

*Proof.* Let  $(\Lambda V, d)$  be a minimal Sullivan model of  $X$ . Suppose that  $V$  is concentrated in even degree. Then  $d = 0$ . Therefore  $H^*(\Lambda V, d) = \Lambda V$  is either concentrated in degree 0 or is not null for an infinite sequence of degrees. By hypothesis, we have excluded these two cases. Therefore  $\dim V^{odd} \geq 1$ .

Let  $x_1, x_2, \dots, x_m, y, x_{m+1}, \dots$  be a basis of  $V$  ordered by degree where  $y$  denotes the first generator of odd degree ( $m \geq 0$ ). For all  $1 \leq i \leq m$ ,  $dx_i \in \Lambda x_{<i}$ . But  $dx_i$  is of odd degree and  $\Lambda x_{<i}$  is concentrated in even degree. So  $dx_i = 0$ . Since  $dy \in \Lambda x_{\leq m}$ ,  $dy$  is equal to a polynomial  $P(x_1, \dots, x_m)$  which belongs to  $\Lambda^{\geq 2}(x_1, \dots, x_m)$ .

Consider  $(\Lambda V \otimes \Lambda sV, \delta)$ , the Sullivan model of  $X^{S^1}$ , given by Theorem 4.4. We have  $\forall 1 \leq i \leq m$ ,  $\delta(sx_i) = -sdx_i = 0$  and  $\delta(sy) = -sdy \in \Lambda^{\geq 1}(x_1, \dots, x_m) \otimes \Lambda^1(sx_1, \dots, sx_m)$ . Therefore, since  $sx_1, \dots, sx_m$  are all of odd degree,  $\forall p \geq 0$ ,

$$\delta(sx_1 \dots sx_m (sy)^p) = \pm sx_1 \dots sx_m p \delta(sy) (sy)^{p-1} = 0.$$

For all  $p \geq 0$ , the cocycle  $sx_1 \dots sx_m (sy)^p$  gives a non trivial cohomology class in  $H^*(X^{S^1}; \mathbb{Q})$ , since by Remark 4.5, the image of this cohomology class in  $H^*(\Omega X; \mathbb{Q}) \cong \Lambda V$  is different from 0.  $\square$

### 6.3 Dimension of $V^{odd} \geq 2$

In this section, we show the following proposition:

**Proposition 6.7.** *Let  $X$  be a simply connected space such that  $H^*(X; \mathbb{Q})$  is of finite (total) dimension and requires at least two generators. Let  $(\Lambda V, d)$  be the minimal Sullivan model of  $X$ . Then  $\dim V^{odd} \geq 2$ .*

**Property 6.8.** (Koszul complexes) Let  $A$  be a graded algebra. Let  $z$  be a central element of even degree of  $A$  which is not a divisor of zero. Then we have a quasi-isomorphism of dgas

$$(A \otimes \Lambda sz, d) \xrightarrow{\cong} A/z.A \quad a \otimes 1 \mapsto a, a \otimes sz \mapsto 0,$$

where  $d(a \otimes 1) = 0$  and  $d(a \otimes sz) = (-1)^{|a|} az$  for all  $a \in A$ .

*Proof of Proposition 6.7 (following (2)  $\Rightarrow$  (3) of p. 214 of [6]).* As we saw in the proof of Theorem 6.6, there is at least one generator  $y$  of odd degree, that is  $\dim V^{odd} \geq 1$ . Suppose that there is only one. Let  $x_1, x_2, \dots, x_m, y, x_{m+1}, \dots$  be a basis of  $V$  ordered by degree ( $m \geq 0$ ).

First case:  $dy = 0$ . If  $m \geq 1$ ,  $dx_1 = 0$ . If  $m = 0$ ,  $dx_1 \in \Lambda^{\geq 2}(y) = \{0\}$  and therefore again  $dx_1 = 0$ . Suppose that for  $n \geq 1$ ,  $x_1^n$  is a coboundary. Then  $x_1^n = d(yP(x_1, \dots)) = yd(P(x_1, \dots))$  where  $P(x_1, \dots)$  is a polynomial in the  $x_i$ 's. But this is impossible since  $x_1^n$  does not belong to the ideal generated by  $y$ .

Therefore for all  $n \geq 1$ ,  $x_1^n$  gives a non trivial cohomology class in  $H^*(X)$ . But  $H^*(X)$  is finite dimensional.

Second case:  $dy \neq 0$ . In particular  $m \geq 1$ . Since  $dy$  is a non zero polynomial,  $dy$  is not a zero divisor, so by Property 6.8, we have a quasi-isomorphism of cdgas

$$\Lambda(x_1, \dots, x_m, y) \xrightarrow{\cong} \Lambda(x_1, \dots, x_m)/(dy).$$

Consider the push out in the category of cdgas

$$\begin{array}{ccc} \Lambda(x_1, \dots, x_m, y) & \longrightarrow & \Lambda(x_1, \dots, x_m, y, x_{m+1}, \dots), d \\ \cong \downarrow & & \downarrow \\ \Lambda(x_1, \dots, x_m)/(dy) & \longrightarrow & \Lambda(x_1, \dots, x_m)/(dy) \otimes \Lambda(x_{m+1}, \dots), \bar{d} \end{array}$$

Since  $\Lambda(x_1, \dots, x_m)/(dy) \otimes \Lambda(x_{m+1}, \dots)$  is concentrated in even degrees,  $\bar{d} = 0$ . Since the top arrow is a Sullivan relative model and the left arrow is a quasi-isomorphism, the right arrow is also a quasi-isomorphism ([5, Lemma 14.2], or more generally the category of cdgas over  $\mathbb{Q}$  is a Quillen model category). Therefore the algebra  $H^*(X)$  is isomorphic to  $\Lambda(x_1, \dots, x_m)/(dy) \otimes \Lambda(x_{m+1}, \dots)$ . If  $m \geq 2$ ,  $\Lambda(x_1, \dots, x_m)/(dy)$  and so  $H^*(X)$  is infinite dimensional. If  $m = 1$ , since  $\Lambda x_1/(dy)$  is generated by only one generator, we must have another generator  $x_2$ . But  $\Lambda(x_1)/(dy) \otimes \Lambda(x_2, \dots)$  is also infinite dimensional.  $\square$

## 6.4 Proof of Vigué-Poirrier-Sullivan theorem

**Lemma 6.9.** [17, Proposition 4] *Let  $A$  be a dga over any field such that the multiplication by a cocycle  $x$  of any degree  $A \rightarrow A$ ,  $a \mapsto xa$  is injective (Our example will be  $A = (\Lambda V, d)$  and  $x$  a non-zero element of  $V$  of even degree such that  $dx = 0$ ). If the Betti numbers  $b_n = \dim H^n(A)$  of  $A$  are bounded then the Betti numbers  $b_n = \dim H^n(A/xA)$  of  $A/xA$  are also bounded.*

*Proof.* Since  $H^n(xA) \cong H^{n-|x|}(A)$ , the short exact sequence of complexes

$$0 \rightarrow xA \rightarrow A \rightarrow A/xA \rightarrow 0$$

gives the long exact sequence in homology

$$\dots \rightarrow H^n(A) \rightarrow H^n(A/xA) \rightarrow H^{n+1-|x|}(A) \rightarrow \dots$$

Therefore  $\dim H^n(A/xA) \leq \dim H^n(A) + \dim H^{n+1-|x|}(A)$   $\square$

*Proof of Vigué-Poirrier-Sullivan theorem (Theorem 6.1).* Let  $(\Lambda V, d)$  be the minimal Sullivan model of  $X$ . Let  $(\Lambda V \otimes \Lambda sV, \delta)$  be the Sullivan model of  $X^{S^1}$  given by Theorem 4.4. From Proposition 6.7, we know that  $\dim V^{odd} \geq 2$ . Let  $x_1, x_2, \dots, x_m, y, x_{m+1}, \dots, x_n, z = x_{n+1}, \dots$  be a basis of  $V$  ordered by degrees where  $x_1, \dots, x_n$  are of even degrees and  $y, z$  are of odd degrees. Consider the

commutative diagram of cdgas where the three rectangles are push outs

$$\begin{array}{ccccc}
\Lambda(x_1, \dots, x_n) & \longrightarrow & (\Lambda V, d) & \longrightarrow & (\Lambda V \otimes \Lambda sV, \delta) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{Q} & \longrightarrow & \Lambda(y, z, \dots) & \longrightarrow & (\Lambda(y, z, \dots) \otimes \Lambda sV, \bar{\delta}) \\
& & \downarrow & & \downarrow \\
& & \mathbb{Q} & \longrightarrow & (\Lambda sV, 0)
\end{array}$$

Note that by Remark 4.5, the differential on  $\Lambda sV$  is 0.

For all  $1 \leq j \leq n+1$ ,

$$\delta x_j = dx_j \in \Lambda^{\geq 2}(x_{<j}, y) \subset \Lambda^{\geq 1}(x_{<j}) \otimes \Lambda y.$$

Therefore

$$\delta(sx_j) = -s\delta x_j \in \Lambda x_{<j} \otimes \Lambda^1 sx_{<j} \otimes \Lambda y + \Lambda^{\geq 1}(x_{<j}) \otimes \Lambda^1 sy.$$

Since  $(sx_1)^2 = \dots = (sx_{j-1})^2 = 0$ , the product

$$sx_1 \dots sx_{j-1} \delta(sx_j) \in \Lambda^{\geq 1}(x_{<j}) \otimes \Lambda^1 sy.$$

So  $\forall 1 \leq j \leq n+1$ ,  $sx_1 \dots sx_{j-1} \bar{\delta}(sx_j) = 0$ . In particular  $sx_1 \dots sx_n \bar{\delta}(sz) = 0$ . Similarly, since  $dy \in \Lambda^{\geq 2} x_{\leq m}$ ,  $sx_1 \dots sx_m \delta(sy) = 0$  and so  $sx_1 \dots sx_n \bar{\delta}(sy) = 0$ . By induction,  $\forall 1 \leq j \leq n$ ,  $\bar{\delta}(sx_1 \dots sx_j) = 0$ . In particular,  $\bar{\delta}(sx_1 \dots sx_n) = 0$ . So finally, for all  $p \geq 0$  and all  $q \geq 0$ ,  $\bar{\delta}(sx_1 \dots sx_n (sy)^p (sz)^q) = 0$ . The cocycles  $sx_1 \dots sx_n (sy)^p (sz)^q$ ,  $p \geq 0$ ,  $q \geq 0$ , give linearly independent cohomology classes in  $H^*(\Lambda(y, z, \dots) \otimes \Lambda sV, \bar{\delta})$  since their images in  $(\Lambda sV, 0)$  are linearly independent.

For all  $k \geq 0$ , there is at least  $k+1$  elements of the form  $sx_1 \dots sx_n (sy)^p (sz)^q$  in degree  $|sx_1| + \dots + |sx_n| + k \cdot \text{lcm}(|sy|, |sz|)$  (just take  $p = i \cdot \text{lcm}(|sy|, |sz|) / |sy|$  and  $q = (k-i) \cdot \text{lcm}(|sy|, |sz|) / |sz|$  for  $i$  between 0 and  $k$ ). Therefore the Betti numbers of  $H^*(\Lambda(y, z, \dots) \otimes \Lambda sV, \bar{\delta})$  are unbounded.

Suppose that the Betti numbers of  $(\Lambda V \otimes \Lambda sV, \delta)$  are bounded. Then by Lemma 6.9 applied to  $A = (\Lambda V \otimes \Lambda sV, \delta)$  and  $x = x_1$ , the Betti numbers of the quotient cdga  $(\Lambda(x_2, \dots) \otimes \Lambda sV, \bar{\delta})$  are bounded. By continuing to apply Lemma 6.9 to  $x_2, x_3, \dots, x_n$ , we obtain that the Betti numbers of the quotient cdga  $(\Lambda(y, z, \dots) \otimes \Lambda sV, \bar{\delta})$  are bounded. But we saw just above that they are unbounded.  $\square$

## 7 Further readings

In this last section, we suggest some further readings that we find appropriate for the student.

In [1, Chapter 19], one can find a very short and gentle introduction to rational homotopy that the reader should compare to our introduction.



In this introduction, we have tried to explain that rational homotopy is a functor which transforms homotopy pullbacks of spaces into homotopy pushouts of cdgas. Therefore after our introduction, we advise the reader to look at [8], a more advanced introduction to rational homotopy, which explains the model category of cdgas.

The canonical reference for rational homotopy [5] is highly readable.

In the recent book [6], you will find many geometric applications of rational homotopy. The proof of Vigué-Poirrier-Sullivan theorem we give here, follows more or less the proof given in [6].

We also like [16] recently reprinted because it is the only book where you can find the Quillen model of a space: a differential graded Lie algebra representing its rational homotopy type (instead of a commutative differential graded algebra as the Sullivan model).

## References

- [1] Raoul Bott and Loring W. Tu, *Differential forms in algebraic topology*, Graduate Texts in Mathematics, vol. 82, Springer-Verlag, New York, 1982.
- [2] William Browder, *Torsion in H-spaces*, Ann. of Math. (2) **74** (1961), 24–51.
- [3] Jean-Luc Brylinski, *Loop spaces, characteristic classes and geometric quantization*, Progress in Mathematics, vol. 107, Birkhäuser Boston Inc., Boston, MA, 1993.
- [4] A. Dold, *Lectures on algebraic topology*, reprint of the 1972 ed., Classics in Mathematics, Springer-Verlag, Berlin, 1995.
- [5] Y. Félix, S. Halperin, and J.-C. Thomas, *Rational homotopy theory*, Graduate Texts in Mathematics, vol. 205, Springer-Verlag, 2000.
- [6] Yves Félix, John Oprea, and Daniel Tanré, *Algebraic models in geometry*, Oxford Graduate Texts in Mathematics, vol. 17, Oxford University Press, Oxford, 2008.
- [7] S. Halperin and M. Vigué-Poirrier, *The homology of a free loop space*, Pacific J. Math. **147** (1991), no. 2, 311–324.
- [8] Kathryn Hess, *Rational homotopy theory: a brief introduction*, Interactions between homotopy theory and algebra, Contemp. Math., vol. 436, Amer. Math. Soc., Providence, RI, 2007, pp. 175–202.
- [9] D. Husemoller, *Fibre bundles*, third ed., Graduate Texts in Mathematics, no. 20, Springer-Verlag, New York, 1994.
- [10] J. D. S. Jones, *Cyclic homology and equivariant homology*, Invent. Math. **87** (1987), no. 2, 403–423.
- [11] L. Menichi, *The cohomology ring of free loop spaces*, Homology Homotopy Appl. **3** (2001), no. 1, 193–224.
- [12] P. Selick, *Introduction to homotopy theory*, Fields Institute Monographs, vol. 9, Amer. Math. Soc., 1997.
- [13] Jean-Pierre Serre, *Homologie singulière des espaces fibrés. Applications*, Ann. of Math. (2) **54** (1951), 425–505.

- [14] Edwin H. Spanier, *Algebraic topology*, Springer-Verlag, New York, 1981, Corrected reprint.
- [15] D. Sullivan, *Differential forms and the topology of manifolds*, Manifolds—Tokyo 1973 (Proc. Internat. Conf., Tokyo, 1973), Univ. Tokyo Press, Tokyo, 1975, pp. 37–49.
- [16] D. Tanré, *Homotopie rationnelle: Modèles de Chen, Quillen, Sullivan*, Lecture Notes in Mathematics, vol. 1025, Springer-Verlag, Berlin-New York, 1983.
- [17] M. Vigué-Poirrier and D. Sullivan, *The homology theory of the closed geodesic problem*, J. Differential Geometry **11** (1976), no. 4, 633–644.

## Index

$A_{PL}$ , 9

cdga, 2

cofibration, 5

dga, 2

graded

algebra, 2

vector space, 1

Hochschild homology, 16, 21

indecomposables, 7

Koszul complexes, 22

Poincaré

serie, 20

quasi-isomorphism, 2

s like operator suspension, 7

Serre

spectral sequence, 21

Sullivan model

minimal, 6

of a cdga, 6

relative, 5