STRING TOPOLOGY OF CLASSIFYING SPACES

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Dedicated to Micheline Vigué-Poirrier for her 60th birthday

Abstract. Let G be a finite group or a compact connected Lie group and let BG be its classifying space. Let $\mathcal{L}BG := map(S^1, BG)$ be the free loop space of BG i.e. the space of continuous maps from the circle S^1 to BG. The purpose of this paper is to study the singular homology $H_*(\mathcal{L}BG)$ of this loop space. We prove that when taken with coefficients in a field the homology of $\mathcal{L}BG$ is a homological conformal field theory. As a byproduct of our main theorem, we get a Batalin-Vilkovisky algebra structure on the cohomology $H^*(\mathcal{L}BG)$. We also prove an algebraic version of this result by showing that the Hochschild cohomology $HH^*(S_*(G), S_*(G))$ of the singular chains of G is a Batalin-Vilkovisky algebra.

AMS Classification : 18D50, 55N91, 55P35, 55P48, 55R12, 55R35, 55R40, 57R56, 58D29, 81T40, 81T45

Key words : free loop space, Hochschild cohomology, props, string topology, topological field theories

Introduction

Popularized by M. Atiyah and G. Segal [4, 61], topological quantum field theories, more generally quantum field theories and their conformal cousins have entered with success the toolbox of algebraic topologists.

Recently they appeared in a fundamental way in the study of the algebraic and differential topology of loop spaces. Let M be a compact, closed, oriented d-dimensional manifold. Let $\mathcal{L}M$ be the free loop space of M. By definition $\mathcal{L}M$ is the space of continuous maps of the circle into M. In their foundational paper "String topology" [12] M. Chas and D. Sullivan introduced a new and very rich algebraic structure on the singular homology $H_*(\mathcal{L}M, \mathbb{Z})$ and its circle equivariant version $H_*^{S^1}(\mathcal{L}M, \mathbb{Z})$. This is a fascinating generalization to higher dimensional manifolds of W. Goldman's bracket [31] which lives on the 0-th space $H_0^{S^1}(\mathcal{L}\Sigma, \mathbb{Z})$ of the equivariant homology of the loops of a closed oriented surface Σ (the free homotopy classes of curves in Σ). Thanks to further works of M. Chas and D. Sullivan [13] 2-dimensional topological field theories, because they encode the ways strings can interact, became the classical algebraic apparatus to understand string topology operations (see also R. Cohen and V. Godin's paper [16]).
Let us also mention that on the geometric side this theory is closely related to Floer homology of the cotangent bundle of M and symplectic field theory. On its more

[0]The first author is supported in part by ANR grant JCJC06 OBTH.

algebraic side the subject relates to Hochschild cohomology and various generalizations of the Deligne's conjecture.

More recently the theory was successfully extended to topological orbifolds [46] and topological stacks [8]. K. Gruher and P. Salvatore also studied a Pro-spectrum version of string topology for the classifying space of a compact Lie group [34]. When twisted topological complex K-theory is applied to this Pro-spectrum, the Pro-cohomology obtained is related to Freed-Hopkins-Teleman theory of twisted K-theory and the Verlinde algebra [35]. With these last developments string topology enters the world of equivariant topology. In this paper we explore the string topology of the classifying space of finite groups and of connected compact Lie groups, advocating that string topology could be applied very naturally in this setting.
Our main theorem is about the field theoretic properties of $\mathcal{L}BG$. We prove that the homology of $\mathcal{L}BG$ is a homological conformal field theory. Let us notice that V. Godin [30] proved an analoguous result for $\mathcal{L}M$, but the techniques used in this setting (fat graphs, embeddings) are completely transverse to those of this paper.

Acknowledgment: The authors would like to thank all the people who helped us in this paper: Michael Crabb, Yves Félix, Jérome Scherer, especially Jean-Claude Thomas and Nathalie Wahl for their helpful comments, Craig Westerland for explaining the second author the cactus action on $H_*(\mathcal{L}M)$ [15, 2.3.1].

1. Plan of the paper and Results

Our main intuition behind all these field theoretic structures was completely algebraic. The second author proved in [52] that the Hochschild cohomology of a symmetric Frobenius algebra is a Batalin-Vilkovisky algebra. The group ring $\mathbb{k}[G]$ of a finite group G is one of the classical example of symmetric Frobenius algebras (Example 48 2)). Therefore the Hochschild cohomology of $\mathbb{k}[G]$ with coefficients in its dual, $HH^*(\mathbb{k}[G];\mathbb{k}[G]^\vee)$, is a Batalin-Vilkovisky algebra (See the beginning of Section 9 for more details). As we know from results of Burghelea-Fiederowicz [11] and Goodwillie [32] that the Hochschild cohomology $HH^*(\mathbb{k}[G];\mathbb{k}[G]^\vee)$ is isomorphic as $\mathbb{k}$-modules to $H^*(\mathcal{L}BG;\mathbb{k})$. Therefore, we obtain

Inspirational Theorem. *Let G be a finite group. Then the singular cohomology with coefficients in any commutative ring $\mathbb{k}$, $H^*(LBG;\mathbb{k})$, is a Batalin-Vilkovisky algebra.*

The question of the geometric incarnation of this structure is natural and will be partially answered in this paper. Note that this Batalin-Vilkovisky algebra is highly non-trivial. For example, the underlying algebra, the Hochschild cohomology ring $HH^*(\mathbb{k}[G];\mathbb{k}[G])$ is studied and computed in some cases in [62].

Section 2 In order to build our "stringy" operations on $\mathcal{L}BG$ we use correspondences and Umkehr maps :
transfer maps for dealing with finite groups,
integration along the fiber for compact Lie groups.
We recall the basic definitions and properties of these maps.

Section 3 As the preceding section this one is expository. We recall the concepts of quantum field theories as axiomatized by Atiyah and Segal. In order to play with these algebraic structures we use the notion of props. The Segal prop of Riemann surfaces and its associated props is described in some details.

Section 4, **section 5**, **section 6**, **section 7** In these sections we build evaluation products for a propic action of the homology of the Segal prop on the homology of the free loop spaces of various groups and topological groups. The aim of these sections is to give a proof of the following theorem.

Main Theorem. (Theorems 44 and 33)
(1) Let G be a discrete finite group or
(2) Let G be a connected topological group such that its singular homology $H_*(G,\mathbb{F})$ with coefficient in a field is finite dimensional.
Then the singular homology of $\mathcal{L}BG$ taken with coefficients in a field, $H_*(\mathcal{L}BG;\mathbb{F})$, is an homological conformal field theory.

The condition (2) in the main theorem deserves some comments. We have originally proved the main theorem for a connected compact Lie group G. The proof being completely homological, the main theorem can be extended for free to (2) which is obviously a weaker condition. Condition (2) is satisfied for finite loop spaces and since the discovery of the Hilton-Roitberg criminal [38] one knows that not every finite loop space is homotopy equivalent to a compact Lie group. When $\mathbb{F}=\mathbb{Q}$ one knows that every odd sphere S^{2n+1} is rationally equivalent to an Eilenberg-MacLane space $K(\mathbb{Q},2n+1)$, therefore condition (2) holds for the group $K(\mathbb{Q},2n+1)$. And when $\mathbb{F}=\mathbb{Z}/p$ the condition is satisfied by p-compact groups [20].
In fact for all the groups G satisfying (2), we will show that their singular homology $H_*(G;\mathbb{F})$ is a symmetric Frobenius algebra, since it is a finite dimensional cocommutative connected Hopf algebra (Proposition 52).

This theorem when restricted to the genus zero and operadic part of the prop of riemann surfaces with boundary gives the topological counterpart of our "inspirational theorem".

Corollary 1. *(Particular case of Corollaries 41 and 36) Let $\mathbb{k}$ be any principal ideal domain.*
(1) Let G be a finite group. Then $H^*(\mathcal{L}BG,\mathbb{k})$ is a Batalin-Vilkovisky algebra.
(2) Let G be a connected compact Lie group of dimension d. Then $H^{*+d}(\mathcal{L}BG,\mathbb{k})$ is a Batalin-Vilkovisky algebra.

Note that Behrend, Ginot, Noohi and Xu [9, Theorem 8.2 and Section 10.4] prove independently part (2) of this Corollary [1].
Section 8. We recall some basic facts about Frobenius algebras and Hopf algebras. We prove that the homology of a connected Lie group together with the Pontryagin product is a symmetric Frobenius algebra (Theorem 53), in fact we offer two different proofs. The first is completely algebraic while the second is topological.

Section 9. We extend our inspirational theorem from finite groups to Lie groups (Theorem 54): *Let G be a connected compact Lie group of dimension d. Let $S_*(G)$*

[1]Wrong, only in homology!

be the singular chains of G. The Gerstenhaber algebra structure on the Hochschild cohomology $HH^*(S_*(G), S_*(G))$ extends to a Batalin-Vilkovisky algebra. There is an isomorphism of vector spaces $HH^*(S_*(G), S_*(G)) \cong H^{*+d}(\mathcal{L}BG)$.]

Section 10. In this section, we define (Theorem 65) a string bracket on the S^1-equivariant cohomology $H^*_{S^1}(\mathcal{L}BG)$ when G is a group satisfying the hypotheses of our main theorem. We also define (Theorem 67) a Lie bracket on the cyclic cohomology $HC^*(S_*(G))$ of $S_*(G)$ when G is a finite group or a connected compact Lie group.

Sections 8, 9 and 10 can be read independently from the rest of the paper. The interested reader might first consider the mod 2 version of Theorem 33. Indeed over $\mathbb{F}_2$, there is no sign and orientation issues for integration along the fiber.

2. Wrong way maps

An *Umkehr map* or wrong way map is a map $f_!$ in homology related to an original continuous map $f : X \to Y$ which reverses the arrow. Umkehr maps can also be considered in cohomology and some of them are refined to stable maps.
A typical example is given when one considers a continuous map $f : M^m \to N^n$ between two oriented closed manifolds. Then using Poincaré duality one defines the associated Umkehr, Gysin, wrong way, surprise or transfer map (depending on your prefered name)

$$f_! : H_*(N^n) \to H_{*+m-n}(M^m).$$

In this paper we will deal with two types of Umkehr maps : transfer and integration along the fiber, both types of Umkehr maps being associated to fibrations.
In the next two sections we review their constructions and in a third section we give a list of their common properties. We refer the reader to chapter 7 of J. C. Becker and D. H. Gottlieb's paper [7] for a nice survey on Umkehr maps.

2.1. **Transfer maps.**

2.1.1. *Transfer for coverings.* Let $p : E \twoheadrightarrow B$ be a covering. Following [37, Beginning of Section 1.3], we don't require that a covering is surjective. Suppose that all the fibers, $p^{-1}(b)$, $b \in B$ are of finite cardinal. As pointed by [2, p. 100], "there is no need to assume that they all have the same cardinal if B is not connected". For example, in [2, (4.3.4)], Adams considers the example of an injective covering p with 1-point fibers and 0-point fibers. Then one can define a map of spectra [2, Construction 4.1.1]

$$\tau_p : \Sigma^\infty B_+ \longrightarrow \Sigma^\infty E_+$$

where $\Sigma^\infty X_+$ denotes the suspension spectrum of the topological space X with a disjoint basepoint added. This map induces in singular homology the transfer map :

$$p_! : H_*(B) \to H_*(E).$$

2.1.2. *Becker-Gottlieb transfer maps.* Let $p : E \to B$ be a fibration over a path-connected base space B. Up to homotopy, we have an unique fiber. Suppose that the fiber F of p has the (stable) homotopy type of a finite complex then one has a stable map

$$\tau_p : \Sigma^\infty B_+ \longrightarrow \Sigma^\infty E_+.$$

Originally constructed by Becker and Gottlieb for smooth fiber bundles with compact fibers [5], they generalize it to fibrations with finite fibres and finite dimensional base spaces [6]. The finiteness condition on the basis has been removed by M. Clapp using duality theory in the category of ex-spaces [14].

2.1.3. *Dwyer's transfer.* At some point we will need to use W. Dwyer's more general version of the transfer [19]. It has the same properties as M. Clapp's transfer (or any other classical version) and is equal to it in the case of the sphere spectrum S^0. Let us consider a ring spectrum R. By definition, a space F is *R-small* [19, Definition 2.2] if the canonical map of spectra

$$map(\Sigma^\infty F_+, R) \wedge \Sigma^\infty F_+ \to map(\Sigma^\infty F_+, R \wedge \Sigma^\infty F_+)$$

is an equivalence. If R is the sphere spectrum S^0, a space F is R-small if F is (stably) homotopy equivalent to a finite CW-complex. If R is an Eilenberg-MacLane spectrum $H\mathbb{Q}$ or $H\mathbb{F}_p$ or a Morava K-theory spectrum $K(n)$, a space F is R-small if $\pi_*(R \wedge \Sigma^\infty F_+)$ is finitely generated as a $\pi_*(R)$-module [19, Exemple 2.15].
Now let $p : E \twoheadrightarrow B$ be a fibration over a path-connected base B and suppose that the fiber F is R-small. Then W. Dwyer has build a transfer map [19, Remark 2.5]

$$\tau_p : R \wedge \Sigma^\infty B_+ \to R \wedge \Sigma^\infty E_+.$$

2.1.4. *Transfer for non-surjective fibrations.* We would like that the Becker-Gottieb (or Dwyer's) transfer extends the transfer for coverings. (Recall that a covering is a fibration).

Let $p : E \twoheadrightarrow B$ be a fibration. We don't require that p is surjective. Suppose that all the fibers, $p^{-1}(b)$, $b \in B$, are (stably) homotopy equivalent to a finite CW-complex. Then we have a Becker-Gottlieb transfer map

$$\tau_p : \Sigma^\infty B_+ \longrightarrow \Sigma^\infty E_+.$$

Proof. Let $\alpha \in \pi_0(B)$. Denote by B_α the path-connected component of B corresponding to α. Let $E_\alpha := p^{-1}(B_\alpha)$ be the inverse image of B_α by p. Let $p_\alpha : E_\alpha \twoheadrightarrow B_\alpha$ the restriction of p to E_α. Either p is surjective or E_α is the empty set $\emptyset$. By pull-back, we have the two weak homotopy equivalence

$$\begin{tikzcd} \coprod_{\alpha\in\pi_0(B)} E_\alpha \arrow[r, "\simeq"] \arrow[d, "\coprod_{\alpha\in\pi_0(B)} p_\alpha"'] & E \arrow[d, "p"] \\ \coprod_{\alpha\in\pi_0(B)} B_\alpha \arrow[r, "\simeq"] & B \end{tikzcd}$$

Since $p_\alpha : E_\alpha \twoheadrightarrow B_\alpha$ is a fibration over a path-connected basis, whose fiber is (stably) homotopy equivalent to a finite CW-complex, we have a Becker-Gottlieb transfer

$$\tau_{p_\alpha} : \Sigma^\infty B_{\alpha+} \longrightarrow \Sigma^\infty E_{\alpha+}.$$

We define the transfer of p by

$$\tau_p := \vee_{\alpha\in\pi_0(B)}\tau_{p_\alpha} : \Sigma^\infty B_+ \simeq \vee_{\alpha\in\pi_0(B)}\Sigma^\infty B_{\alpha+} \longrightarrow \vee_{\alpha\in\pi_0(B)}\Sigma^\infty E_{\alpha+} \simeq \Sigma^\infty E_+.$$

In singular homology, we have the linear map of degree 0

$$p_! := \oplus_{\alpha\in\pi_0(B)} p_{\alpha!} : H_*(B) \cong \oplus_{\alpha\in\pi_0(B)} H_*(B_\alpha) \longrightarrow \oplus_{\alpha\in\pi_0(B)} H_*(E_\alpha) \cong H_*(E).$$

□

2.2. **Integration along the fiber.** If we have a smooth oriented fiber bundle $p : E \to B$, the integration along the fiber F can be defined at the level of the de Rham cochain complex by integrating differential forms on E along F. This defines a map in cohomology

$$p^! : H^*(E, \mathbb{R}) \to H^*(B, \mathbb{R})$$

this was the very first definition of integration along the fiber and it goes back to A. Lichnerowicz [43].
We review some well-known generalizations of this construction, for our purpose we need to work with fibrations over an infinite dimensional basis. As we just need to work with singular homology (at least in this paper), we will use Serre's spectral sequence.

2.2.1. *A spectral sequence version.* ([33, Section 2], [41, Chapter 2, Section 3], [2, p. 106] or [56, Section 4.2.3]) Let $F \hookrightarrow E \overset{p}{\twoheadrightarrow} B$ be a fibration over a path-connected base B. We suppose that the homology of the fiber $H_*(F, \mathbb{F})$ is concentrated in degree less than n and has a top non-zero homology group $H_n(F, \mathbb{F}) \cong \mathbb{F}$. Let us assume that the action of the fundamental group $\pi_1(B)$ on $H_n(F, \mathbb{F})$ induced by the holonomy is trivial. Let ω be a generator of $H_n(F, \mathbb{F})$ i.e. an orientation class. We shall refer to such data as an *oriented fibration*.
Using the Serre spectral sequence, one can define the integration along the fiber as a map

$$p_! : H_*(B) \to H_{*+n}(E).$$

Let us recall the construction, we consider the spectral sequence with local coefficients [55]. As the Serre spectral sequence is concentrated under the n-th line, the filtration on the abutment $H_{l+n}(E)$ is of the form

$$0 = F^{-1} = F^0 = \cdots = F^{l-1} \subset F^l \subset F^{l+1} \subset \cdots \subset F^{l+n} = H_{l+n}(E).$$

As the local coefficients are trivial by hypothesis, the orientation class ω defines an isomorphism of local coefficients $\tau : \mathbb{F} \to \mathcal{H}_n(F_b, \mathbb{F})$. By definition $p_!$ is the composite

$$p_! : H_l(B, \mathbb{F}) \overset{H_l(B;\tau)}{\to} H_l(B, \mathcal{H}_n(F_b, \mathbb{F})) = E^2_{l,n} \twoheadrightarrow E^\infty_{l,n} = \frac{F^l}{F^{l-1}} = F^l \subset H_{l+n}(E, \mathbb{F}).$$

2.3. **Properties of Umkehr maps.** Let us give a list of properties that are satisfied by transfer maps and integration along the fibers. In fact all reasonable notion of Umkehr map must satisfy this Yoga. We write these properties for integration along the fiber taking into account the degree shifting, we let the reader do the easy translation for transfer maps.

Naturality [41, p. 29]: Consider a commutative diagram

$$\begin{tikzcd} E_1 \arrow[r, "g"] \arrow[d, "p_1"] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "h"] & B_2 \end{tikzcd}$$

where p_1 is a fibration over a path-connected base and p_2 equipped with the orientation class $w_2 \in H_n(F_2)$ is an oriented fibration. Let $f : F_1 \to F_2$ the map induced between the fibers. Suppose that $H_*(f)$ is an isomorphism. Then the fibration p_1

equipped with the orientation class $w_1 := H_n(f)^{-1}(w_2)$ is an oriented fibration and the following diagram commutes

$$\begin{tikzcd} H_{*+n}(E_1) \arrow[r, "H_*(g)"] & H_{*+n}(E_2) \\ H_*(B_1) \arrow[u, "p_1!"] \arrow[r, "H_*(h)"] & H_*(B_2) \arrow[u, "p_2!"'] \end{tikzcd}$$

Proof. With respect to the morphism of groups $\pi_1(h) : \pi_1(B_1) \to \pi_1(B_2)$, $H_n(f) : H_n(F_1) \to H_n(F_2)$ is an isomorphism of $\pi_1(B_1)$-modules. Since $\pi_1(B_2)$ acts trivially on $H_n(F_2)$, $\pi_1(B_1)$ acts trivially on $H_n(F_1)$. By definition of the orientation class w_1 and by naturality of the Serre spectral sequence, the diagram follows. ☐

We will apply the naturality property in the following two cases

Naturality with respect to pull-back: [63, chapter 9.Section 2.Theorem 5 b)] Suppose that we have a pull-back, then f is an homeomorphism.

Naturality with respect to homotopy equivalences: Suppose that g and h are homotopy equivalences, then f is a homotopy equivalence.

Composition: Let $f : X \twoheadrightarrow Y$ be an oriented fibration with path-connected fiber F_f and orientation class $w_f \in H_m(F_f)$. Let $g : Y \twoheadrightarrow Z$ be a second oriented fibration with path-connected fiber F_g and orientation class $w_g \in H_n(F_g)$. Then the composite $g \circ f : X \to Z$ is an oriented fibration with path-connected fiber $F_{g\circ f}$. By naturality with respect to pull-back, we obtain an oriented fibration $f' : F_{g\circ f} \twoheadrightarrow F_g$ with orientation class $w_f \in H_m(F_f)$. By definition, the orientation class of $g \circ f$ is $w_{g\circ f} := f'_!(w_g) \in H_{n+m}(F_{g\circ f})$. Then we have the commutative diagram

$$\begin{tikzcd} & H_{*+n}(Y) \arrow[dr, "f!"] & \\ H_*(Z) \arrow[ur, "g!"] \arrow[rr, "(g\circ f)!"] & & H_{*+m+n}(X). \end{tikzcd}$$

Product: Let $p : E \twoheadrightarrow B$ be an oriented fibration with fiber F and orientation class $w \in H_m(F)$. Let $p' : E' \twoheadrightarrow B'$ be a second oriented fibration with fiber F' and orientation class $w' \in H_n(F')$. Then if one work with homology with field coefficients, $p \times p' : E \times E' \twoheadrightarrow B \times B'$ is a third oriented fibration with fiber $F \times F'$ and orientation class $w \times w' \in H_{m+n}(F \times F')$ and one has for $a \in H_*(B)$ and $b \in H_*(B')$,

$$(p \otimes p')_!(a \otimes b) = (-1)^{|a|n} p_!(a) \otimes p'_!(b).$$

Notice that since $p'_!$ is of degree n, the sign $(-1)^{|a|n}$ agrees with the Koszul rule.

Borel construction: Let G be a topological group acting continuously on two topological spaces E and B, we also suppose that we have a continuous G-equivariant map

$$p : E \to B$$

the induced map on homotopy G-quotients (we apply the Borel functor $EG \times_G -$ to p) is denoted by

$$p_{hG} : E_{hG} \to B_{hG}.$$

We suppose that the action of G on B has a fixed point b and that $p : E \twoheadrightarrow B$ is an oriented fibration with fiber $F := p^{-1}(b)$ and orientation class $w \in H_n(F)$. This fiber F is a sub G-space of E. Then we suppose that the action of G preserves the orientation, to be more precise we suppose that the action of $\pi_0(G)$ on $H_n(F)$ is trivial. Then

$$p_{hG} : E_{hG} \twoheadrightarrow B_{hG}$$

is locally an oriented (Serre) fibration and therefore is an oriented (Serre) fibration [63, Chapter 2.Section 7.Theorem 13] with fiber F and orientation class $w \in H_n(F)$. Note that under the same hypothesis, $p_!$ is $H_*(G)$-linear (Compare with Lemma 56).

2.4. **The yoga of correspondences.** Let us finish this section by an easy lemma on Umkehr maps, once again we give it for integration along the fiber and let the reader translate it for transfers.
We will formulate it in the language of oriented correspondences, the idea of using correspondences in string topology is due to S. Voronov [15, section 2.3.1]. We introduce a category of oriented correspondences denoted by $\mathbf{Corr_{or}}$:
- the objects of our category will be path connected spaces with the homotopy type of a CW-complex
- $Hom_{\mathbf{Corr_{or}}}(X, Y)$ is given by the set of oriented correspondences between X and Y a correspondence will be a sequence of continuous maps :

$$Y \overset{r_2}{\leftarrow} Z \overset{r_1}{\rightarrow} X$$

such that r_1 is an oriented fibration.
Composition of morphisms is given by pull-backs, if we consider two correspondences

$$Y \overset{r_2}{\leftarrow} Z \overset{r_1}{\rightarrow} X$$

and

$$U \overset{r'_2}{\leftarrow} T \overset{r'_1}{\rightarrow} Y$$

then the composition of the correspondences is

$$U \overset{r"_2}{\leftarrow} T \times_Y Z \overset{r"_1}{\rightarrow} X.$$

We have to be a little bit more careful in the definition of morphisms, composition as defined above is not strictly associative (we let the reader fix the details).

Lemma 2. Composition lemma. *The singular homology with coefficients in a field defines a symmetric monoidal functor*

$$\mathbb{H}(-, \mathbb{F}) : \mathbf{Corr_{or}} \longrightarrow \mathbb{F} - \mathbf{vspaces}$$

to a morphism $Y \overset{r_2}{\leftarrow} Z \overset{r_1}{\rightarrow} X$ *it associates* $(r_2)_* \circ (r_1)_! : H_*(X, \mathbb{F}) \to H_{*+d}(Y, \mathbb{F})$ *(where d is the homological dimension of the fiber of r_1).*

Proof. The fact that the functor is monoidal follows from the product property of the integration along the fiber.
Let c and c' be two oriented correspondences the fact that $\mathbb{H}(c' \circ c, \mathbb{F}) = \mathbb{H}(c', \mathbb{F}) \circ$

$\mathbb{H}(c,\mathbb{F})$ follows from an easy inspection of the following diagram

$$\begin{tikzcd}
 & & X \\
 & T\times_Y Z \arrow[ur, "r''_1"] \arrow[r, "f_1"] \arrow[d, "f_2"'] \arrow[dl, "r''_2"'] & Z \arrow[u, "r_1"'] \arrow[d, "r_2"] \\
U & T \arrow[l, "r'_2"] \arrow[r, "r'_1"'] & Y
\end{tikzcd}$$

in fact we have $(r'_2)\circ(r'_1)_!\circ(r_2)_*\circ(r_1)_! = (r'_2)_*\circ(f_2)_*\circ(f_1)_!\circ(r_1)_!$ by the naturality property and by the composition property we have $(f_1)_!\circ(r_1)_! = (r''_1)_!$. □

3. Props and field theories

The aim of this section is to introduce the algebraic notions that encompass the "stringy" operations acting on $\mathcal{L}BG$. This section is mainly expository.

3.1. **props.** We use props and algebras over them as a nice algebraic framework in order to deal with 2-dimensional field theories. We could in this framework use the classical tools from algebra and homological algebra exactly as for algebras over operads.

Definition 3. [50, Definition 54] A *prop* is a symmetric (strict) monoidal category $\mathcal{P}$ [42, 3.2.4] whose set of objects is identified with the set $\mathbb{Z}_+$ of nonnegative numbers. The tensor law on objects should be given by addition of integers $p\otimes q = p+q$. Strict monoidal means that the associativity and neutral conditions are the identity.

We thus have two composition products on morphisms a horizontal one given by the tensor law :

$$-\otimes - : \mathcal{P}(p,q)\otimes\mathcal{P}(p',q') \longrightarrow \mathcal{P}(p+p',q+q'),$$

and a vertical one given by composition of morphisms :

$$-\circ - : \mathcal{P}(q,r)\otimes\mathcal{P}(p,q) \longrightarrow \mathcal{P}(p,r).$$

Example 4. Let V be a fixed vector space. A fundamental example of prop is given by the *endomorphims prop of* V denoted $\mathcal{E}nd_V$. The set of morphisms is defined as $\mathcal{E}nd_V(p,q) = Hom(V^{\otimes p}, V^{\otimes q})$. The horizontal composition product is just the tensor while the vertical is the composition of morphisms.

A morphism of props is a symmetric (strict) monoidal functor [42, 3.2.48] F such that $F(1) = 1$. Let $\mathcal{P}$ be a linear prop i.e. we suppose that $\mathcal{P}$ is enriched in the category of vector spaces (graded, differential graded) and V be a vector space (graded, differential graded,....).

Definition 5. [50, Definition 56] The vector space V is said to be a $\mathcal{P}$*-algebra* if there is a morphism of linear props

$$F : \mathcal{P} \longrightarrow \mathcal{E}nd_V.$$

This means that we have a a family of linear morphisms

$$F : \mathcal{P}(m,n) \to \mathrm{Hom}(V^{\otimes m}, V^{\otimes n})$$

such that

(monoidal) $F(f\otimes g) = F(f)\otimes F(g)$ for $f\in\mathcal{P}(m,n)$ and $g\in\mathcal{P}(m',n')$.

(identity) The image $F(id_n)$ of the identity morphism $id_n \in \mathcal{P}(n,n)$, is equal to the identity morphism of $V^{\otimes n}$.

(symmetry) $F(\tau_{m,n}) = \tau_{V^{\otimes m},V^{\otimes n}}$. Here $\tau_{m,n} : m \otimes n \to n \otimes m$ denotes the natural twist isomorphism of $\mathcal{P}$. And for any graded vector spaces V and W, $\tau_{V,W}$ is the isomorphism $\tau_{V,W} : V \otimes W \to W \otimes V$, $v \otimes w \mapsto (-1)^{|v||w|} w \otimes v$.

(composition) $F(g \circ f) = F(g) \circ F(f)$ for $f \in \mathcal{P}(p,q)$ and $g \in \mathcal{P}(q,r)$.

By adjunction, this morphism determines evaluation products

$$\mu : \mathcal{P}(p,q) \otimes V^{\otimes p} \longrightarrow V^{\otimes q}.$$

Remarks. One can also notice that the normalized singular chain functor sends topological props to props in the category of differential graded modules. If the homology of a topological prop is also a prop, this is not always the case for algebras over props (because props also encode coproducts), one has to consider homology with coefficients in a field. This is the main reason, why in this paper, we have chosen to work over an arbitrary field $\mathbb{F}$.

3.2. **Topological Quantum Field Theories.** Let F_1 and F_2 be two smooth closed oriented n-dimensional manifolds not necessarily path-connected.

Definition 6. [54, p. 201] An *oriented cobordism* from F_1 to F_2 is a n-dimensional smooth compact oriented manifold F not necessarily path-connected with boundary ∂F, equipped with an orientation preserving diffeomorphism φ from the disjoint union $-F_1 \coprod F_2$ to ∂F (The orientation on ∂F being the one induced by the orientation of F).

We call *in-coming boundary* map $in : F_1 \hookrightarrow F$ the composite of the restriction of φ to F_1 and of the inclusion map of ∂F into F. We call *out-coming boundary* map $out : F_2 \hookrightarrow F$ the composite of the restriction of φ to F_2 and of the inclusion map of ∂F into F.

Let F and F' be two oriented cobordisms from F_1 to F_2.

Definition 7. [42, 1.2.17] We say that F and F' are *equivalent* if there is an orientation preserving diffeomorphism ϕ from F to F' such that the following diagram commutes

$$\begin{tikzcd}
 & F \arrow[dd, "\cong"', "\phi"] & \\
F_1 \arrow[ur, "in"] \arrow[dr, "in"'] & & F_2 \arrow[ul, "out"'] \arrow[dl, "out"] \\
 & F' &
\end{tikzcd}$$

Definition 8. [42, 1.3.20] The category of oriented cobordisms $n - Cob$ is the discrete category whose objets are smooth closed oriented, not necessarily path-connected, $n-1$-dimensional manifolds. The morphisms from F_1 to F_2 are the set of equivalent classes of oriented cobordisms from F_1 to F_2. Composition is given by gluing cobordisms. Disjoint union gives n-cob a structure of symmetric monoidal category.

Definition 9. [42, 1.3.32] A n-dimensional Topological Quantum Field Theory (n-TQFT) as axiomatized by Atiyah is a symmetric monoidal functor from the category of oriented cobordisms n-Cob to the category of vector spaces.

3.3. **The linear prop defining 2-TQFTs** $\mathbb{F}[sk(2-Cob)]$**.** We now restrict to $n=2$, i. e. to 2-TQFTs. By [42, 1.4.9], the skeleton of the category $2-Cob$, $sk(2-Cob)$, is the full subcategory of $2-Cob$ whose objects are disjoint union of circles $\coprod_{i=1}^n S^1$, $n\geq 0$. Therefore $sk(2-Cob)$ is a discrete prop. For any set X, denote by $\mathbb{F}[X]$ the free vector space with basis X. By applying the functor $\mathbb{F}[-]$ to $sk(2-Cob)$, we obtain a linear prop $\mathbb{F}[sk(2-Cob)]$. And now a vector space V is a 2-Topological Quantum Field Theory if and only if V is an algebra over this prop $\mathbb{F}[sk(2-Cob)]$.

3.4. **Segal prop of Riemann surfaces** $\mathfrak{M}$**.** ([15, Example 2.1.2], compare with [49, p. 24 and p. 207]) Let p and $q\geq 0$.

A "complex cobordism" from the disjoint union $\coprod_{i=1}^p S^1$ to $\coprod_{i=1}^q S^1$ is a closed complex curve F, not necessarily path connected equipped with two holomorphic embeddings of disjoint union of closed disks into F, $in:\coprod_{i=1}^p D^2\hookrightarrow F$ and $out:\coprod_{i=1}^q D^2\hookrightarrow F$.

Let F and F' be two "complex cobordisms" from $\coprod_{i=1}^p S^1$ to $\coprod_{i=1}^q S^1$.

We say that F and F' are *equivalent* if there is a biholomorphic map ϕ from F to F' such that the following diagram commutes

$$\begin{tikzcd} & F \arrow[dd, "\cong" ', "\phi"] & \\ \coprod_{i=1}^p D^2 \arrow[ur, "in"] \arrow[dr, "in"'] & & \coprod_{i=1}^q D^2 \arrow[ul, "out"'] \arrow[dl, "out"] \\ & F' & \end{tikzcd}$$

The Segal prop $\mathfrak{M}$ is the topological category whose objects are disjoint union of circles $\coprod_{i=1}^n S^1$, $n\geq 0$, identified with non-negative numbers. The set of morphisms from p to q, denoted $\mathfrak{M}(p,q)$, is the set of equivalent classes of "complex cobordisms" from $\coprod_{i=1}^p S^1$ to $\coprod_{i=1}^q S^1$. The moduli space $\mathfrak{M}(p,q)$ is equipped with a topology difficult to define [49, p. 207].

By applying the singular homology functor with coefficients in a field, $H_*(-)$ to the Segal topological prop $\mathfrak{M}$, one gets a graded linear prop $H_*(\mathfrak{M})$. Explicitly

$$H_*(\mathfrak{M})(p,q):=H_*(\mathfrak{M}(p,q)).$$

Definition 10. [15, 3.1.2] A graded vector space V is a *(unital counital) homological conformal field theory* or HCFT for short if V is an algebra over the graded linear prop $H_*(\mathfrak{M})$.

3.5. **The props isomorphism** $\pi_0(\mathfrak{M})\cong sk(2-Cob)$**.** Let

$$\coprod_{i=1}^p D^2 \overset{in}{\hookrightarrow} F \overset{out}{\hookleftarrow} \coprod_{i=1}^q D^2$$

be a "complex cobordism" from $\coprod_{i=1}^p S^1$ to $\coprod_{i=1}^q S^1$. By forgetting the complex structure and removing the interior of the $p+q$ disks, we obtain an oriented cobordism $F-\coprod_{i=1}^{p+q} IntD^2$ from $-\coprod_{i=1}^p S^1$ to $\coprod_{i=1}^q S^1$. Indeed the restrictions of in to $\coprod_{i=1}^p S^1$, $in_{|\coprod_p S^1}$, and the restriction of out to $\coprod_{i=1}^p S^1$, $out_{|\coprod_q S^1}$, are orientation preserving diffeomorphims. By composing $in_{|\coprod_p S^1}$ with a reversing orientation diffeomorphism, $F-\coprod_{i=1}^{p+q} IntD^2$ becomes an oriented cobordism from $\coprod_{i=1}^p S^1$ to $\coprod_{i=1}^q S^1$. Therefore, we have defined a morphism of props $Forget:\mathfrak{M}\to sk(2-Cob)$.

The topology on the moduli spaces $\mathfrak{M}(p,q)$ is defined such that this morphism of props *Forget* is continuous and can be identified with the canonical surjective morphism of topological props $\mathfrak{M} \twoheadrightarrow \pi_0(\mathfrak{M})$ from the Segap prop to the discrete prop obtained by taking its path components.

3.6. **Tillmann prop.** Following U. Tillman's topological approach to the study of the Moduli spaces of complex curves we introduce a topological prop BD homotopy equivalent to a sub prop of Segal prop of Riemann surfaces $\mathfrak{M}$.

For p and q, consider the groupoid $\mathcal{E}(p,q)$ [70, p. 69]. An object of $\mathcal{E}(p,q)$ is an oriented cobordism F from $\amalg_{i=1}^{p} S^1$ to $\amalg_{i=1}^{p} S^1$ (Definition 6). The set of morphisms from F_1 to F_2, is

$$\mathrm{Hom}_{\mathcal{E}(p,q)}(F_1,F_2) := \pi_0 Diff^+(F_1;F_2;\partial)$$

the connected components of orientation preserving diffeomorphisms ϕ that fix the boundaries of F_1 and F_2 pointwise: $\phi\circ in = in$ and $\phi\circ out = out$. Remark that if $F_1 = F_2$ is a connected surface $F_{g,p+q}$, then $\mathrm{Hom}_{\mathcal{E}(p,q)}(F_{g,p+q},F_{g,p+q})$ is the mapping class group

$$\Gamma_{g,p+q} := \pi_0 Diff^+(F_{g,p+q};\partial).$$

In [67] U. Tillman studies the homotopy type of a surface 2-category $\mathcal{S}$. Roughly speaking, the objects of $\mathcal{S}$ are natural numbers representing circles. The enriched set of morphisms from p to q is the category $\mathcal{E}(p,q)$. The composition in $\mathcal{S}$ is the functor induced by gluing $\mathcal{E}(q,r)\times\mathcal{E}(p,q)\to\mathcal{E}(p,r)$. U. Tillmann's surface category $\mathcal{S}$ has the virtue to be a symmetric strict monoidal 2-category where the tensor product is given by disjoint union of cobordisms.

Let $\mathcal{C}$ be a small category. The nerve of $\mathcal{C}$ is a simplicial set $N(\mathcal{C})$. The classifying space of $\mathcal{C}$, $B(\mathcal{C})$, is by definition the geometric realization of this simplicial set, $N(\mathcal{C})$. Applying the classifying space functor to $\mathcal{S}$, one gets a topological symmetric (strict) monoidal category. We thus have a topological prop $\mathcal{BS}$. By definition, $\mathcal{BS}(p,q) := B(\mathcal{E}(p,q))$.

These categories and their higher dimensional analogues have been studied extensively because of their fundamental relationship with conformal field theories and Mumford's conjecture [28, 48].

Recall that if $\mathcal{C}$ is a groupoid, then $\pi_0(B\mathcal{C})$ is the set of isomorphisms classes of $\mathcal{C}$. Therefore the skeleton of the category of oriented cobordisms, $sk(2-Cob(p,q))$ (Section 3.3), is exactly the discrete prop obtained by taking the path-components of the topological prop $\mathcal{BS}$ (Compare with Section 3.5). In particular,

$$sk(2-Cob(p,q)) := \pi_0(\mathcal{BS}(p,q)) = \pi_0(B(\mathcal{E}(p,q)).$$

By considering the skeleton of a groupoid $\mathcal{C}$, we have the homotopy equivalence

$$\coprod_x B(\mathrm{Hom}(x,x)) \overset{\approx}{\to} B\mathcal{C}.$$

where the disjoint union is taken over a set of representatives x of isomorphism classes in $\mathcal{C}$ and $B(\mathrm{Hom}(x,x))$ is the classifying space of the discrete group $\mathrm{Hom}(x,x)$ [21, 5.10]. Therefore the morphism spaces $B\mathcal{S}(p,q)$ have a connected component for each oriented cobordism class F (Definition 7). The connected component corresponding to F has the homotopy type of $B(\pi_0(Diff^+(F,\partial)))$ [67, p. 264].

The cobordism $F_{g,p+q}$ is the disjoint union of its path components:

$$F_{g,p+q} \cong F_{g_1,p_1+q_1} \amalg \cdots \amalg F_{g_k,p_k+q_k}.$$

Here F_{g_i,p_i+q_i} denotes a surface of genus g_i with p_i incoming and q_i outgoing circles target. We have $g=\sum_i g_i$, $p=\sum_i p_i$ and $q=\sum_i q_i$. We suppose that each path component F_{g_i,p_i+q_i} has at least one boundary component. That is, we suppose that $\forall 1\leq i\leq k$, $p_i+q_i\geq 1$. Since a diffeomorphism fixing the boundaries pointwise cannot exchange the path-components of F, we have the isomorphism of topological groups

$$Diff^+(F;\partial)\cong Diff^+(F_{g_1,p_1+q_1};\partial)\times\cdots\times Diff^+(F_{g_k,p_k+q_k};\partial)$$

Again since $p_i+q_i\geq 1$, by [22], the canonical surjection from

$$Diff^+(F_{g_i,p_i+q_i};\partial)\xrightarrow{\approx}\Gamma_{g_i,p_i+q_i}:=\pi_0 Diff^+(F_{g_i,p_i+q_i};\partial).$$

with the discrete topology is a homotopy equivalence. Therefore the canonical surjection:

$$Diff^+(F;\partial)\xrightarrow{\approx}\pi_0 Diff^+(F;\partial)$$

is also a homotopy equivalence: Earle and Schatz result [22] extends to a non-connected surface F if each component has at least one boundary component. Therefore we have partially recovered the following proposition:

Proposition 11. [68, 3.2] *For $p\in\mathbb{Z}^+$ and $q\in\mathbb{Z}^+$, define the collection of topological spaces*

$$BD(p,q):=\coprod_{F_{p+q}} BDiff^+(F;\partial).$$

Here the disjoint union is taken over a set of representatives F_{p+q} of the oriented cobordism classes from $\amalg_{i=1}^p S^1$ to $\amalg_{i=1}^q S^1$ (Definition 7). This collection BD of spaces forms a topological prop up to homotopy (it is a prop in the homotopy category of spaces). If we consider only cobordisms F whose path components have at least one outgoing-boundary component, i. e. $q_i\geq 1$, (this is the technical condition of [67, p. 263]*), the resulting three sub topological props of BD, Tillmann's prop $B\mathcal{S}$ and Segal prop $\mathfrak{M}$ are all homotopy equivalent.*

3.7. **Non-unital and non-counital homological conformal field theory.**

Definition 12. (Compare with definition 10) A graded vector space V is a *unital non-counital* homological conformal field theory if V is an algebra over the graded linear prop obtained by applying singular homology to one of the three sub topological props defined in the previous Proposition.

In [30], Godin uses the term homological conformal field theory *with positive boundary* instead of unital non-counital.

If instead, we consider only cobordisms F whose path components have at least one in-boundary component, i. e. $p_i\geq 1$, we will say that we have a *counital non-unital* homological conformal field theory.

In this paper, we will deal mainly with *non-unital non-counital* homological conformal field theory: this is when we consider only cobordisms F whose path components have at least one in-boundary component and also at least one outgoing-boundary component, i. e. $\forall 1\leq i\leq k$, $p_i\geq 1$ and $q_i\geq 1$.

4. Definition of the operations

The goal of this section is to define the evaluation products of the propic action of the homology of the Segal prop.

4.1. **The definition assuming Propositions 14, 15 and Theorem 17.** Let $F_{g,p+q}$ be an oriented cobordism from $\amalg_{i=1}^{p} S^1$ to $\amalg_{i=1}^{q} S^1$. Let k be its number of path components and let g be the sum of the genera of its path components. Let

$$\chi(F) = 2k - 2g - p - q$$

be its Euler characteristic. Let $Diff^+(F;\partial)$ be the group of orientation preserving diffeomorphisms that fix the boundaries pointwise.

Definition 13. Let X be a simply-connected space such that its pointed loop homology $H_*(\Omega X)$ is a finite dimensional vector space. Denote by d the top degree such that $H_d(\Omega X)$ is not zero.

Suppose that every path component of $F_{g,p+q}$ has at least one in-boundary component and at least one outgoing-boundary component. Then we can define the *evaluation product* associated to $F_{g,p+q}$.

$$\mu(F) : H_*(BDiff^+(F;\partial)) \otimes H_*(\mathcal{L}X)^{\otimes p} \to H_*(\mathcal{L}X)^{\otimes q}.$$

It is a linear map of degre $-d\chi(F)$.

Proof of Definition 13. Let $F_{g,p+q}$ be an oriented cobordism (not necessarily path-connected) with $p+q$ boundary components equipped with a given ingoing map

$$in : \partial_{in} F_{g,p+q} = \amalg_{i=1}^{p} S^1 \hookrightarrow F_{g,p+q}$$

and an outgoing map

$$out : \partial_{out} F_{g,p+q} = \amalg_{i=1}^{q} S^1 \hookrightarrow F_{g,p+q}.$$

These two maps are cofibrations. The cobordism $F_{g,p+q}$ is the disjoint union of its path components:

$$F_{g,p+q} \cong F_{g_1,p_1+q_1} \amalg \cdots \amalg F_{g_k,p_k+q_k}.$$

Recall that we suppose that $\forall 1 \leq i \leq k$, $p_i \geq 1$ and $q_i \geq 1$. Therefore, by Proposition 14, one obtain that the cofibre of in, $F/\partial_{in}F$, is homotopy equivalent to the wedge $\vee_{-\chi(F)} S^1$ of $-\chi(F) = 2g+p+q-2k$ circles. When we apply the mapping space $map(-,X)$ to the cofibrations in and out, one gets the two fibrations

$$map(in, X) : map(F_{g,p+q}, X) \twoheadrightarrow map(\partial_{in} F_{g,p+q}, X) \cong \mathcal{L}X^{\times p}$$

and

$$map(out, X) : map(F_{g,p+q}, X) \twoheadrightarrow map(\partial_{out} F_{g,p+q}, X) \cong \mathcal{L}X^{\times q}.$$

The fiber of the continuous map $map(in,X)$ is the pointed mapping space $map_*(F/\partial_{in}F, X)$ and is therefore homotopy equivalent to the product of pointed loop spaces $\Omega X^{-\chi(F)}$.

Since $H_*(\Omega X)$ is a Hopf algebra and is finite dimensional, by [65, Proof of Corollary 5.1.6 2)], $H_*(\Omega X)$ is a Frobenius algebra: i. e. there exists an isomorphism

$$H_*(\Omega X) \overset{\cong}{\to} H_*(\Omega X)^\vee \cong H^*(\Omega X)$$

of left $H_*(\Omega X)$-modules. Since $H_*(\Omega X)$ is concentrated in degree between 0 and d, and $H_0(\Omega X)$ and $H_d(\Omega X)$ are not trivial vector spaces, this isomorphism is of lower degree $-d$:

$$H_p(\Omega X) \overset{\cong}{\to} H_{d-p}(\Omega X)^\vee \cong H^{d-p}(\Omega X).$$

In particular, since X is simply connected, $H_d(\Omega X) \cong H_0(\Omega X)^\vee \cong H^0(\Omega X) \cong \mathbb{F}$ is of dimension 1.

Therefore the homology of the fibre of $map(in,X)$ is concentrated in degree less or equal than $-d\chi(F)$ and $H_{-d\chi(F)}(map_*(F/\partial_{in}F, X)) \cong \mathbb{F}$ is also of dimension 1.

By Proposition 15, we have that in fact $map(in, X) : map(F_{g,p+q}, X) \twoheadrightarrow \mathcal{L}X^{\times p}$ is an oriented fibration.

We set $D_{g,p+q} := Diff^+(F_{g,p+q}, \partial)$. We also denote the Borel construction

$$\mathcal{M}_{g,p+q}(X) := (map(F_{g,p+q}, X))_{hD_{g,p+q}}.$$

Applying the Borel construction $(-)_{hD_{g,p+q}}$ to the fibrations $map(in, X)$ and $map(out, X)$ yields the following two fibrations

$$\rho_{in} := map(in, X)_{hD_{g,p+q}} : \mathcal{M}_{g,p+q}(X) \longrightarrow BD_{g,p+q} \times \mathcal{L}X^{\times p}.$$

$$\rho_{out} : \mathcal{M}_{g,p+q}(X) \overset{map(out,X)_{hD_{g,p+q}}}{\longrightarrow} BD_{g,p+q} \times \mathcal{L}X^{\times q} \overset{proj_2}{\longrightarrow} LX^{\times q}.$$

Here $proj_2$ is the projection on the second factor.

By Theorem 17, $\pi_0(D_{g,p+q})$ acts trivially on $H_{-d\chi(F)}(map_*(F/\partial_{in}F, X))$. Under this condition, the Borel construction $(-)_{hD_{g,p+q}}$ preserves oriented (Serre) fibration. Therefore ρ_{in} is also an oriented fibration with fibre $map_*(F/\partial_{in}F, X)$. After choosing an orientation class (Section 11)

$$\omega_F \in H_{-d\chi(F)}(map_*(F/\partial_{in}F, X)),$$

we have a well defined integration along the fiber map for ρ_{int}:

$$\rho_{in}! : H_*(BD_{g,p+q} \times \mathcal{L}X^{\times p}) \longrightarrow H_{*-d\chi_F}(\mathcal{M}_{q,p+p}(X)).$$

By composing with $H_*(\rho_{out}) : H_*(\mathcal{M}_{g,p+q}(X)) \longrightarrow H_*(\mathcal{L}X^{\times q})$, one gets a map

$$\mu(F_{g,p+q}) : H_l(BD_{g,p+q}) \otimes H_{m_1}(\mathcal{L}X) \otimes \cdots \otimes H_{m_p}(\mathcal{L}X) \to H_{l+m_1+\cdots+m_p-d\chi_F}(\mathcal{L}X^{\times q}).$$

As we restrict ourself either to homology with coefficients in a field, one finally gets an evaluation product of degree $-d\chi_F = d(2g + p + q - 2k)$

$$\boxed{\mu(F_{g,p+q}) : H_*(BD_{g,p+q}) \otimes H_*(\mathcal{L}X)^{\otimes p} \to H_*(\mathcal{L}X)^{\otimes q}}$$

□

4.2. Orientability of the fibration $map(in, X) : map(F, X) \twoheadrightarrow \mathcal{L}X^{\times p}$.

Proposition 14. *Let $F_{g,p+q}$ be the path-connected compact oriented surface of genus g with p incoming boundary circles and q outgoing boundary circles. Denote by $F/\partial_{in}F$, the cofibre of*

$$in : \partial_{in}F_{g,p+q} = \amalg_{i=1}^p S^1 \hookrightarrow F_{g,p+q}.$$

If $p \geq 1$ and $q \geq 1$ then $F/\partial_{in}F$ is homotopy equivalent to the wedge $\vee_{-\chi(F)}S^1$ of $-\chi(F) = 2g + p + q - 2$ circles.

Proof. We first show that $F/\partial_{in}F$ and the wedge of $-\chi(F)$ circles have the same homology. If $p \geq 1$ then $\tilde{H}_0(F/\partial_{in}F) = \{0\}$. By excision and Poincaré duality [37, Theorem 3.43], if $q \geq 1$,

$$\tilde{H}_2(F/\partial_{in}F) \cong H_2(F, \partial_{in}F) \cong H^0(F, \partial_{out}F) \cong \tilde{H}^0(F/\partial_{out}F) = \{0\}.$$

Using the long exact sequence associated to the pair $(\partial_{in}F, F)$, we obtain that $\chi(H_*(F)) = \chi(H_*(\partial_{in}F)) + \chi(H_*(F, \partial_{in}F))$. Therefore the Euler characteristic $\chi(H_*(F))$ of $H_*(F)$ is equal to the Euler characteristic $\chi(H_*(F, \partial_{in}F))$ of $H_*(F, \partial_{in}F)$. Therefore $\tilde{H}_1(F/\partial_{in}F) = \tilde{H}_*(F/\partial_{in}F)$ is of dimension $-\chi(F)$.

We show that $F/\partial_{in}F$ is homotopy equivalent to a wedge of circles. The surface $F_{g,n-1}$, $n \geq 1$, can be constructed from a full polygon with $4g + n - 1$ sides by identifying pairs of the $4g$ edges [37, p. 5]. Therefore the surface with one more boundary component, $F_{g,n}$, $n \geq 1$, is the mapping cylinder of the attaching map from S^1 to $F^{(1)}_{g,n-1}$, the 1-skeleton of $F_{g,n-1}$ (in the case $n = 1$, see [37, Example 1B.14]). Therefore any surface with boundary deformation retracts onto a connected graph [37, Example 1B.2].

By filling the p incoming circles S^1 by p incoming disks D^2, we have the push-out

$$\begin{tikzcd} \coprod_{i=1}^p S^1 \arrow[r, "in"] \arrow[d] & F_{g,p+q} \arrow[d] \\ \coprod_{i=1}^p D^2 \arrow[r] & F_{g,0+q}. \end{tikzcd}$$

So we obtain that

$$\frac{F_{g,p+q}}{\coprod_{i=1}^p S^1} \cong \frac{F_{g,0+q}}{\coprod_{i=1}^p D^2}$$

is homotopy equivalent to the mapping cone of an application from p distinct points to a path-connected graph. Therefore $\frac{F_{g,p+q}}{\coprod_{i=1}^p S^1}$ is homotopy equivalent to a path-connected graph and therefore to a wedge of circles [37, Example 0.7]. □

Now we prove that the fibration is oriented by studying the action of the fundamental group of the basis on the homology of the fiber.

Proposition 15. *Let $F_{g,p+q}$ be a path-connected cobordism from $\coprod_{i=1}^p S^1$ to $\coprod_{i=1}^q S^1$. Let X be a simply connected space such that $H_*(\Omega X, \mathbb{F})$ is finite dimensional. If $p \geq 1$ and $q \geq 1$ then the fibration obtained by restriction to the in-boundary components*

$$map(in, X) : map(F_{g,p+q}, X) \twoheadrightarrow \mathcal{L}X^{\times p}$$

is $H_*(-,\mathbb{F})$-oriented.

Proof. The key for the proof of Proposition 14 was that a surface $F_{g,p+q}$ with boundary is homotopy equivalent to a graph. For the proof of this Proposition, we are more precise: we use a special kind of graphs, called the Sullivan Chord diagrams. If $p \geq 1$ and $q \geq 1$ then the surface $F_{g,p+q}$ is homotopy equivalent to a Sullivan's chord diagram $c_{g,p+q}$ of type $(g; p, q)$.

A Sullivan's Chord diagram $c_{g,p+q}$ of type $(g; p, q)$ [16, Definition 2] consists of an union of p disjoint circles together with the disjoint union of path-connected trees. The endpoints of the trees are joined at distincts points to the p circles.

Denote by $\sigma(c)$ this set of path-connected trees. The endpoints of the trees lying on the p circles are called the circular vertices and the set of circular vertices of $c_{g,p+q}$ is denoted $v(c)$. For each tree $v \in \sigma(c)$, denoted by $\mu(v)$ the endpoints of the tree v which are on the p circles. We have the disjoint union $v(c) = \coprod_{v\in\sigma(c)} \mu(v)$. Therefore, with the notations introduced (that follows the notations of Cohen and Godin [16, Section 2]), we have the push-out

$$\begin{tikzcd} v(c) = \coprod_{v\in\sigma(c)} \mu(v) \arrow[r] \arrow[d] & \coprod_{i=1}^p S^1 \arrow[d, "in"] \\ \coprod_{v\in\sigma(c)} v \arrow[r] & c_{g,p+q} \end{tikzcd}$$

Up to a homotopy equivalence between $F_{g,p+q}$ and $c_{g,p+q}$, the p circles in $c_{g,p+q}$ represent the incoming boundary components of $F_{g,p+q}$, i. e. we have the commutative triangle

$$\begin{tikzcd} \coprod_{i=1}^{p} S^1 \arrow[rr, "in"] \arrow[dr, "in"'] & & c_{g,p+q} \\ & F_{g,p+q} \arrow[ur, "\approx"'] & \end{tikzcd}$$

Let $\#\mu(v)$, $\#v(c)$ and $\#\sigma(c)$ denote the cardinals of the sets $\mu(v)$, $v(c)$ and $\sigma(c)$. By applying the mapping space $map(-, X)$, we have the commutative diagram where the square is a pull-back.

(16)
$$\begin{tikzcd} map(F_{g,p+q}, X) \arrow[dr, "map(in,X)"'] & map(c_{g,p+q}, X) \arrow[l, "\approx"'] \arrow[r] \arrow[d, "map(in,X)"] & \prod_{v\in\sigma(c)} map(v, X) \arrow[d] \\ & (\mathcal{L}X)^{\times p} \arrow[r] & \prod_{v\in\sigma(c)} X^{\#\mu(v)} = X^{\#v(c)} \end{tikzcd}$$

As $X^{\#v(c)}$ is simply connected, the fibration

$$\prod_{v\in\sigma(c)} map(v, X) \twoheadrightarrow \prod_{v\in\sigma(c)} X^{\#\mu(v)} = X^{\#v(c)}$$

is oriented. As orientation is preserved by pull-back and homotopy equivalence, the fibration $map(in, X) : map(F_{g,p+q}, X) \twoheadrightarrow (\mathcal{L}X)^{\times p}$ is also oriented: the action of $\pi_1(\mathcal{L}X^{\times p})$ preserves the orientation class $\omega_{F,\theta} \in H_{-d\chi(F)}(map_*(F/\partial_{in}F, X))$.

Remark that using Sullivan Chord diagrams, we can give a second but more complicated proof of Proposition 14: Since the path-connected trees $v \in \sigma(c)$ are contractile and since the cardinal of $\mu(v)$, $\#\mu(v)$ is always not zero, the cofiber of the cofibration $\mu(v) \hookrightarrow v$ is homotopy equivalent to a wedge $\vee_{\#\mu(v)-1}S^1$ of $\#\mu(v)-1$ circles. Therefore the fiber of the fibration

$$\prod_{v\in\sigma(c)} map(v, X) \twoheadrightarrow \prod_{v\in\sigma(c)} X^{\#\mu(v)} = X^{\#v(c)}$$

is homotopy equivalent to the product $\prod_{v\in\sigma(c)} \Omega X^{\#\mu(v)-1} = \Omega X^{\#v(c)-\#\sigma(c)}$. By Mayer-Vietoris long exact sequence, we have the additivity formula for the Euler characteristic

$$\chi(F_{g,p+q}) = \chi(c_{g,p+q}) = \chi(\coprod_{i=1}^{p} S^1) + \chi(\coprod_{v\in\sigma(c)} v) - \chi(\coprod_{v\in\sigma(c)} \mu(v)) = 0 + \#\sigma(c) - \#v(c).$$

Since fibers are preserved by pull-backs, we recover that the fiber of $map(in, X)$ is homotopy equivalent to $\Omega X^{-\chi(F_{g,p+q})}$. $\square$

Examples :

1) In order to illustrate the proof of the preceding proposition let us consider the fundamental example of the pair of pants P, viewed as a cobordism between two ingoing circles and one outgoing circle. In this particular case one has $c_{0,2+1} = O-O$. One replaces the space $map(P, X)$ by $map(O-O, X)$ and as $\sharp\sigma(O-O) = 1$ and $\sharp v(O-O) = 2$ we have to deal with the pull-back diagram

$$\begin{tikzcd} map(O-O, X) \arrow[r] \arrow[d, "map(in,X)"'] & map(I, X) \arrow[d, "(ev_0,ev_1)"] \\ (\mathcal{L}X)^{\times 2} \arrow[r, "ev_0\times ev_0"'] & X^{\times 2}. \end{tikzcd}$$

The product on $H_*(\mathcal{L}X)$ is the composite

$$H_*(\mathcal{L}X\times\mathcal{L}X)\overset{map(in,X)!}{\to}H_{*+d}(map(O-O,X))\overset{H_*(out)}{\to}H_{*+d}(\mathcal{L}X).$$

2) Let us consider again the pair of pants P, viewed this time as a cobordism between one ingoing circle and two outgoing circles. In this case, $c_{0,1+2}=\emptyset$ and we have the pull-back diagram

$$\begin{tikzcd} map(\emptyset,X) \arrow[r] \arrow[d, "map(in,X)"'] & map(I,X) \arrow[d, "(ev_0,ev_1)"] \\ \mathcal{L}X \arrow[r, "(ev_0,ev_{1/2})"'] & X^{\times 2}. \end{tikzcd}$$

The coproduct on $H_*(\mathcal{L}X)$ is the composite

$$H_*(\mathcal{L}X)\overset{map(in,X)!}{\to}H_{*+d}(map(\emptyset,X))\overset{H_*(out)}{\to}H_{*+d}(\mathcal{L}X\times\mathcal{L}X).$$

Our pull-back square (16) is the same as the one considered in [16, section 3], except that we did not collapse the trees $v\in\sigma(c)$ (or the "ghost edges" of $c_{g,p+q}$ with the terminology of [16, section 3]), since we want oriented fibrations and Cohen and Godin wanted embeddings in order to have shriek maps.

For example, the two Sullivan's Chord diagrams $O-O$ and $\emptyset$ give both after collapsing the unique edge of their unique tree (or the unique "ghost edge") the famous figure eight ∞, considered by Chas and Sullivan.

Therefore up to this collapsing, our product on $H_*(\mathcal{L}X)$ is defined as the Chas-Sullivan loop product on $H_*(\mathcal{L}M)$ for manifolds. Our coproduct on $H_*(\mathcal{L}X)$ is defined as the Cohen-Godin loop coproduct on $H_*(\mathcal{L}M)$.

4.3. **Orientability of the fibration ρ_{in}.**

Theorem 17. *Let $F_{g,p+q}$ be a path-connected cobordism from $\coprod_{i=1}^{p}S^1$ to $\coprod_{i=1}^{q}S^1$. Assume that $p\geq 1$ and that $q\geq 1$ and let $\chi(F)=2-2g-p-q$ be the Euler characteristic of $F_{g,p+q}$. Let X be a simply connected space such that $H_*(\Omega X)$ is a finite dimensional vector space. Denote by d the top degree such that $H_d(\Omega X)\neq\{0\}$. Then the action of $Diff^+(F,\partial)$ on $H_{-d\chi(F)}(map_*(F/\partial_{in}F,X))$ is trivial.*

To prove this Theorem, we will need the following Propositions 19 and 20.

Property 18. (Compare with [17, Lemma 1 p. 1176]) Consider a commutative diagram of exact sequences of abelian groups

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "a"] & B \arrow[r] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & D \arrow[r] \arrow[d, "d"] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & D \arrow[r] & 0. \end{tikzcd}$$

If A, C and D are free and finitely generated then B is also free and finitely generated and the determinants of the vertical morphisms satisfy the equality

$$\det(a)\det(c)=\det(b)\det(d).$$

Proof. First consider the case $A=\{0\}$ where we have short exact sequences. A splitting $D\to C$, a basis of B and a basis of D gives a basis of C where the matrix of c is a triangular by block matrix of the form $\begin{pmatrix} b & ? \\ 0 & d\end{pmatrix}$. Therefore $\det(c)=\det(b)\det(d)$.

The general case follows by splitting $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 0$ into two short exact sequences. $\square$

Proposition 19. *Let $F_{g,p+q}$ be a path-connected cobordism from $\coprod_{i=1}^{p} S^1$ to $\coprod_{i=1}^{q} S^1$ with $p \geq 0$ and $q \geq 0$. Let $f \in Homeo^+(F, \partial)$ be an homeomorphism from $F_{g,p+q}$ to $F_{g,p+q}$ that fixes the boundary pointwise and preserve the orientation (for example if $f \in Diff^+(F, \partial)$). Then the induced isomorphism in singular cohomology*

$$H^1(f;\mathbb{Z}) : H^1(F, \partial_{in}F;\mathbb{Z}) \to H^1(F, \partial_{in}F;\mathbb{Z})$$

is of determinant $+1$.

Remark that if $p+q \geq 1$, by the exact sequence

$$0 = H_2(F_{g,p+q};\mathbb{Z}) \to H_2(F_{g,p+q}, \coprod_{p+q} S^1;\mathbb{Z}) \to H_1(\coprod_{p+q} S^1;\mathbb{Z}),$$

continous maps $f : (F_{g,p+q}, \coprod_{p+q} S^1) \to (F_{g,p+q}, \coprod_{p+q} S^1)$ that fix the boundary $\coprod_{p+q} S^1$ are automatically orientation preserving.

Proof. By filling the boundary of $F_{g,p+q}$ with $p+q$ closed disks D^2, we have the push out

$$\begin{tikzcd}
\coprod_{p+q} S^1 \arrow[r] \arrow[d] & F_{g,p+q} \arrow[d] \arrow[dr, "f"] & \\
\coprod_{p+q} D^2 \arrow[r] \arrow[dr, "id"'] & F_g \arrow[dr, dotted, "\tilde{f}"] & F_{g,p+q} \arrow[d] \\
& \coprod_{p+q} D^2 \arrow[r] & F_g
\end{tikzcd}$$

Denote by $Homeo^+(F_g, \coprod_{p+q} D^2)$ the group of orientation preserving homeomorphisms from F_g to F_g that fix the $p+q$ embedded disks pointwise. By universal property of push outs, any $f \in Homeo^+(F, \partial)$ can be extended to an unique $\tilde{f} \in Homeo^+(F_g, \coprod_{p+q} D^2)$. Since

$$H_2(F_g) \cong H_2(F_g, \coprod_{p+q} D^2) \cong H_2(F_{g,p+q}, \coprod_{p+q} S^1),$$

$H_2(\tilde{f}) : H_2(F_g) \to H_2(F_g)$ is the identity if and only if

$$H_2(f) : H_2(F_{g,p+q}, \coprod_{p+q} S^1) \to H_2(F_{g,p+q}, \coprod_{p+q} S^1)$$

is also the identity. Therefore $\tilde{f}$ preserves the orientation if and only if f also preserves the orientation. Since the restriction of $\tilde{f}$ to $F_{g,p+q}$ is f, the two groups $Homeo^+(F_g, \coprod_{p+q} D^2)$ and $Homeo^+(F, \partial)$ are isomorphic. Observe that with diffeomorphisms instead of homeomorphisms, we don't have an isomorphism between $Diff^+(F_g, \coprod_{p+q} D^2)$ and $Diff^+(F, \partial)$ althought the two groups are usually identified [56, p. 169].

Since $H^*(\tilde{f})$ preserves the cup product $\cup$ and the fundamental class $[F_g] \in Hom_{\mathbb{Z}}(H^2(F_g), \mathbb{Z})$, $H^1(\tilde{f})$ preserves the symplectic bilinear form on $H^1(F_g) \cong \mathbb{Z}^{2g}$ defined by $< a, b > := [F_g](a \cup b)$ for a and $b \in H^1(F_g)$. Since symplectic automorphisms are of determinant $+1$, $H^1(\tilde{f})$ is of determinant $+1$ and the well known [56, Definition of Torelli group] case $p = q = 0$ is proved.

We now prove by induction that $\forall i$ such that $0 \leq i \leq q$, $H^1(\tilde{f}_{|F_{g,0+i}}) : H^1(F_{g,0+i}) \to H^1(F_{g,0+i})$ is of determinant $+1$. This will prove the case $p = 0$. If $i \geq 1$, consider the commutative diagram of long exact sequences

$$\begin{tikzcd}
0 = \tilde{H}^0(S^1) \arrow[r] & H^1(F_{g,0+i-1}) \arrow[r] \arrow[d, "H^1(\tilde{f}_{|F_{g,0+i-1}})"'] & H^1(F_{g,0+i}) \arrow[r] \arrow[d, "H^1(\tilde{f}_{|F_{g,0+i}})"'] & H^1(S^1) \arrow[r] \arrow[d, "H^1(f_{|S^1})"'] & H^2(F_{g,0+i-1}) \arrow[r] \arrow[d, "H^2(\tilde{f}_{|F_{g,0+i-1}})"'] & H^2(F_{g,0+i}) = \\
0 = \tilde{H}^0(S^1) \arrow[r] & H^1(F_{g,0+i-1}) \arrow[r] & H^1(F_{g,0+i}) \arrow[r] & H^1(S^1) \arrow[r] & H^2(F_{g,0+i-1}) \arrow[r] & H^2(F_{g,0+i}) =
\end{tikzcd}$$

Since the restriction of f to S^1, $f_{|S^1}$, is the identity morphism, $H^2(\tilde{f}_{|F_{g,0+i-1}})$ is also the identity morphism. Therefore by Property 18, the determinant of $H^1(\tilde{f}_{|F_{g,0+i}})$ is equal to the determinant of $H^1(\tilde{f}_{|F_{g,0+i-1}})$ which is by induction hypothesis $+1$. This finishes the induction.

Suppose now that $p \geq 1$ and $q \geq 0$. Consider the commutative diagram of long exact sequences

$$\begin{tikzcd}
0 = H^0(F_{g,0+q}, \coprod_p D^2) \arrow[d] & H^0(F_{g,0+q}, \coprod_p D^2) = 0 \arrow[d] \\
H^0(F_{g,0+q}) \arrow[r, "H^0(\tilde{f}_{|F_{g,0+q}})"] \arrow[d] & H^0(F_{g,0+q}) \arrow[d] \\
H^0(\coprod_p D^2) \arrow[r, "H^0(\tilde{f}_{|\coprod_p D^2})"] \arrow[d] & H^0(\coprod_p D^2) \arrow[d] \\
H^1(F_{g,0+q}, \coprod_p D^2) \arrow[r, "H^1(\tilde{f}_{|F_{g,0+q}})"] \arrow[d] & H^1(F_{g,0+q}, \coprod_p D^2) \arrow[d] \\
H^1(F_{g,0+q}) \arrow[r, "H^1(\tilde{f}_{|F_{g,0+q}})"] \arrow[d] & H^1(F_{g,0+q}) \arrow[d] \\
0 = H^1(\coprod_p D^2) & H^1(\coprod_p D^2) = 0
\end{tikzcd}$$

Since the restriction $\tilde{f}_{|\coprod_p D^2}$ is the identity, $H^0(\tilde{f}_{|F_{g,0+q}})$ is also the identity. We have proved that

$$H^1(\tilde{f}_{|F_{g,0+q}}) : H^1(F_{g,0+q}) \to H^1(F_{g,0+q})$$

is of determinant $+1$. Therefore, by Property 18,

$$H^1(\tilde{f}_{|F_{g,0+q}}) : H^1(F_{g,0+q}, \coprod_p D^2) \to H^1(F_{g,0+q}, \coprod_p D^2)$$

is also of determinant $+1$. Since $H^1(F_{g,p+q}, \partial_{in} F) \cong H^1(F_{g,0+q}, \coprod_p D^2)$, the determinants of

$$H^1(f) : H^1(F_{g,p+q}, \partial_{in} F) \to H^1(F_{g,p+q}, \partial_{in} F)$$

and

$$H^1(\tilde{f}_{|F_{g,0+q}}) : H^1(F_{g,0+q}, \coprod_p D^2) \to H^1(F_{g,0+q}, \coprod_p D^2)$$

are equal. $\square$

Proposition 20. *Let $n \geq 0$ be a non-negative integer. Let $h : \vee_n S^1 \overset{\approx}{\to} \vee_n S^1$ be a pointed homotopy equivalence from a wedge of n circles to itself. Let X be a simply connected space such that $H_*(\Omega X)$ is finite dimensional. Let d be the top degree such that $H_d(\Omega X) \neq \{0\}$. Then in the top degre,*

$$H_{dn}(map_*(h, X); \mathbb{F}) : H_{dn}(map_*(\vee_n S^1, X); \mathbb{F}) \to H_{dn}(map_*(\vee_n S^1, X); \mathbb{F})$$

is the multiplication by $(det H_1(h; \mathbb{Z}))^d$, the d-th power of the determinant of

$$H_1(h; \mathbb{Z}) : H_1(\vee_n S^1 \mathbb{Z}) = \mathbb{Z}^n \to H_1(\vee_n S^1 \mathbb{Z}) = \mathbb{Z}^n.$$

Proof. If $n = 0$ then h is the identity map and the proposition follows since $H_1(h; \mathbb{Z}) : \{0\} \to \{0\}$ is of determinant $+1$. Denote by $h \vee S^1 : (\vee_n S^1) \vee S^1 \overset{\approx}{\to} (\vee_n S^1) \vee S^1$ the homotopy equivalence extending h and the identity of S^1. Note that det $H_1(h \vee S^1; \mathbb{Z}) =$ det $H_1(h; \mathbb{Z})$. Since $H_d(\Omega X)$ is of dimension 1 and $map_*(h \vee S^1, X) = map_*(h, X) \times \Omega X$, by naturality of Kunneth theorem, we have the commutative diagram of vector spaces

$$\begin{tikzcd} H_{dn} map_*(\vee_n S^1, X) \arrow[r, "\cong"] \arrow[d, "H_{dn}(map_*(h,X))"'] & H_{dn} map_*(\vee_n S^1, X) \otimes H_d(\Omega X) \arrow[r, "\cong"] \arrow[d, "H_{dn}(map_*(h,X)) \otimes H_d(\Omega X)"'] & H_{d(n+1)} map_*(\vee_{n+1} S^1, X) \arrow[d, "H_{d(n+1)}(map_*(h \vee S^1, X))"'] \\ H_{dn} map_*(\vee_n S^1, X) \arrow[r, "\cong"'] & H_{dn} map_*(\vee_n S^1, X) \otimes H_d(\Omega X) \arrow[r, "\cong"'] & H_{d(n+1)} map_*(\vee_{n+1} S^1, X) \end{tikzcd}$$

where the horizontal morphisms are isomorphisms. So if $H_{d(n+1)}(map_*(h \vee S^1, X); \mathbb{F})$ is the multiplication by (det $H_1(h \vee S^1; \mathbb{Z}))^d$, then $H_{dn}(map_*(h, X); \mathbb{F})$ is also the multiplication by (det $H_1(h \vee S^1; \mathbb{Z}))^d =$ (det $H_1(h; \mathbb{Z}))^d$. Therefore if the proposition is proved for an integer $n + 1$, the proposition is also proved for the previous integer n.

It remains to prove the proposition for large n: we assume now that $n \geq 3$.

Let $\text{aut}_* \vee_n S^1$ be the monoid of pointed self equivalences of the wedge of n circles. Recall that $\pi_1(\vee_n S^1)$ is isomorphic to the free group F_n on n letters. Denote by AutF_n the groups of automorphisms of F_n. Since $\vee_n S^1$ is a $K(\pi, 1)$, by [37, Chapter 1 Proposition 1B.9] and Whitehead theorem, the morphism of groups $\pi_1(-) : \pi_0(\text{aut}_* \vee_n S^1) \to \text{Aut}F_n$ sending the homotopy class $[h]$ of $h \in \text{aut}_* \vee_n S^1$ to the group automorphism $\pi_1(h)$, is an isomorphism of groups.

The monoid $\text{aut}_* \vee_n S^1$ acts on the right on $map_*(\vee_n S^1, X)$ by composition. Therefore, we have a morphism of monoids

$$map_*(-, X) : \text{aut}_* \vee_n S^1 \to map\left(map_*(\vee_n S^1, X), map_*(\vee_n S^1, X)\right)^{op}.$$

Here *op* denotes the opposite monoid. Passing to homology, we have a morphism of groups into the opposite of the general linear group of the $\mathbb{F}$-vector space $H_{dn} map_*(\vee_n S^1, X)$.

$$H_{dn}(map_*(-, X)) : \pi_0(\text{aut}_* \vee_n S^1) \to \text{GL}_{\mathbb{F}}(H_{dn} map_*(\vee_n S^1, X))^{op}.$$

Since the vector space $H_{dn}(map_*(\vee_n S^1, X))$ is of dimension 1, the trace map

$$\text{Trace} : \text{GL}_{\mathbb{F}}(H_{dn} map_*(\vee_n S^1, X)) \xrightarrow{\cong} \mathbb{F} - \{0\}$$

is an isomorphism of abelian groups. Denote by $(-)_{Ab}$ the Abelianisation functor from groups to abelian groups. Since $\mathbb{F} - \{0\}$ is an abelian group, by universal property of abelianisation, we have a commutative diagram of groups

$$\begin{tikzcd}
\mathrm{Aut}F_n \arrow[d] & \pi_0(\mathrm{aut}_* \vee_n S^1) \arrow[l, "\pi_1(-)"', "\cong"] \arrow[r, "H_{dn}(map_*(-,X))"] & \mathrm{GL}_{\mathbb{F}}(H_{dn}map_*(\vee_n S^1, X))^{op} \arrow[d, "\cong"', "Trace"] \\
(\mathrm{Aut}F_n)_{Ab} \arrow[rr, dotted] & & \mathbb{F}-\{0\}
\end{tikzcd} \tag{21}$$

The abelianisation of F_n, $(F_n)_{Ab}$ is isomorphic to $\mathbb{Z}^n$. Denote by

$$\mathrm{Ab}(-) : \mathrm{Aut}F_n \to \mathrm{Aut}\left((F_n)_{Ab}\right) = \mathrm{GL}_n(\mathbb{Z}),$$

the morphism of groups sending $f : F_n \to F_n$ to $f_{Ab} : \mathbb{Z}^n \to \mathbb{Z}^n$. Since $\mathbb{Z}/2\mathbb{Z}$ is an abelian group, by universal property of abelianisation, we have a commutative diagram of groups

$$\begin{tikzcd}
\mathrm{Aut}F_n \arrow[r, two heads, "\mathrm{Ab(-)}"] \arrow[d] & \mathrm{GL}_n(\mathbb{Z}) \arrow[d, "Det"] \\
(\mathrm{Aut}F_n)_{Ab} \arrow[r, dotted, "\cong"'] & \mathbb{Z}/2\mathbb{Z}
\end{tikzcd} \tag{22}$$

Since both the determinant map $\mathrm{Det} : \mathrm{GL}_n(\mathbb{Z}) \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$ and $\mathrm{Ab}(-) : \mathrm{Aut}F_n \twoheadrightarrow \mathrm{GL}_n(\mathbb{Z})$ are surjective [47, Chapter I.Proposition 4.4], the morphism $(\mathrm{Aut}F_n)_{Ab} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$ is surjective. By [69, Section 5.1], since $n \geq 3$, the abelianisation of $\mathrm{Aut}F_n$, $(\mathrm{Aut}F_n)_{Ab}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Therefore this surjective morphism of abelian groups $(\mathrm{Aut}F_n)_{Ab} \overset{\cong}{\twoheadrightarrow} \mathbb{Z}/2\mathbb{Z}$ is in fact an isomorphism.

The composition $\pi_0(\mathrm{aut}_* \vee_n S^1) \overset{\pi_1(-)}{\to} \mathrm{Aut}F_n \overset{\mathrm{Ab}(-)}{\to} \mathrm{GL}_n(\mathbb{Z})$ coincides with the morphism of groups $H_1(-;\mathbb{Z}) : \pi_0(\mathrm{aut}_* \vee_n S^1) \to \mathrm{GL}_n(\mathbb{Z})$ sending the homotopy class $[h]$ of $h \in \mathrm{aut}_* \vee_n S^1$ to the isomorphism of abelian groups $H_1(h;\mathbb{Z}) : \mathbb{Z}^n \overset{\cong}{\to} \mathbb{Z}^n$. Therefore putting side by side diagram (21) and diagram (22), we obtain the commutative diagram of groups

$$\begin{tikzcd}
\mathrm{GL}_n(\mathbb{Z}) \arrow[d, "Det"'] & \pi_0(\mathrm{aut}_* \vee_n S^1) \arrow[l, "H_1(-;\mathbb{Z})"'] \arrow[r, "H_{dn}(map_*(-,X))"] & \mathrm{GL}_{\mathbb{F}}(H_{dn}map_*(\vee_n S^1, X))^{op} \arrow[d, "\cong"', "Trace"] \\
\mathbb{Z}/2\mathbb{Z} \arrow[rr, dotted, "i"'] & & \mathbb{F}-\{0\}
\end{tikzcd} \tag{23}$$

We want to compute the morphism of groups $i : \mathbb{Z}/2\mathbb{Z} \to \mathbb{F}-\{0\}$. In order to distinguish the circles in $\vee_{i=1}^n S^1$, we consider $\vee_{i=1}^n S^1$ as the quotient space $\frac{S^1\times\{1,\cdots,n\}}{*\times\{1,\cdots,n\}}$: an element of $\vee_{i=1}^n S^1$ is the class of (x,i), $x \in S^1$, $i \in \{1,\cdots,n\}$.

Any permutation $\sigma \in \Sigma_n$ induces a pointed homeomorphism $\sigma\cdot - : \vee_{i=1}^n S^1 \overset{\cong}{\to} \vee_{i=1}^n S^1$ defined by $\sigma\cdot(x,i) = (x,\sigma(i))$. The matrix of $H_1(\sigma\cdot -;\mathbb{Z})$, which is the image of $\sigma\cdot -$ by the morphism $H_1(-;\mathbb{Z}) : \pi_0(\mathrm{aut}_* \vee_n S^1) \to \mathrm{GL}_n(\mathbb{Z})$, is the permutation matrix $M_\sigma := (m_{ij})_{i,j}$ defined by

$$m_{\sigma(i)j} = \begin{cases} 1 & \text{Si } j = i, \\ 0 & \text{Si } j \neq i. \end{cases}$$

Since the determinant of the permutation matrix M_σ is the sign of σ, $\varepsilon(\sigma)$,

$$\mathrm{Det} \circ H_1(-;\mathbb{Z})(\sigma\cdot -) = \varepsilon(\sigma).$$

On the other hand, $H_*(map_*(\sigma\cdot -, X)) : H_*(\Omega X)^{\otimes n} \to H_*(\Omega X)^{\otimes n}$ maps $w_1\otimes\cdots\otimes w_n$ to $\pm w_{\sigma(1)} \otimes \cdots \otimes w_{\sigma(n)}$ where $\pm$ is the Koszul sign. In particular, let τ be the transposition (12) of Σ_n, $H_{dn}(map_*(\tau \cdot -, X)) : H_d(\Omega X)^{\otimes n} \to H_d(\Omega X)^{\otimes n}$ is the multiplication by $(-1)^d$. Since the sign of τ is -1, by commutativity of diagram (23), we have

$$i(-1) = i(\varepsilon(\tau)) = i \circ \mathrm{Det} \circ H_1(-;\mathbb{Z})(\tau \cdot -) = \mathrm{Trace} H_{dn}(map_*(\tau \cdot -, X)) = (-1)^d.$$

□

Proof of Theorem 17. Let $f \in Diff^+(F, \partial)$. Since the diffeomorphism f fixes the boundary pointwise, f induces a pointed homeomorphism $\bar{f} : F/\partial_{in}F \overset{\cong}{\to} F/\partial_{in}F$. The action of f on $map_*(F/\partial_{in}F, X)$ is given by $map_*(\bar{f}, X)$. Since $p \geq 1$ and $q \geq 1$, by Proposition 14, there exists a pointed homotopy equivalence $g : \vee_{-\chi(F)} S^1 \overset{\approx}{\to} F/\partial_{in}F$. Consider a pointed homotopy equivalence $h : \vee_{-\chi(F)} S^1 \overset{\approx}{\to} \vee_{-\chi(F)} S^1$ such that $g \circ h$ is (pointed) homotopic to $\bar{f} \circ g$. Since $map_*(-, X)$ preserves pointed homotopies, we have the commutative diagram of $\mathbb{F}$-vector spaces

$$\begin{tikzcd}
H_{-d\chi(F)}(map_*(F/\partial_{in}F, X)) \arrow[r, "H_{-d\chi(F)}(map_*(\bar{f},X))"] \arrow[d, "\cong", "H_{-d\chi(F)}(map_*(g,X))"'] & H_{-d\chi(F)}(map_*(F/\partial_{in}F, X)) \arrow[d, "H_{-d\chi(F)}(map_*(g,X))", "\cong"'] \\
H_{-d\chi(F)}(map_*(\vee_{-\chi(F)} S^1, X)) \arrow[r, "H_{-d\chi(F)}(map_*(h,X))"'] & H_{-d\chi(F)}(map_*(\vee_{-\chi(F)} S^1, X))
\end{tikzcd}.$$

Since by Proposition 20 applied to h with $n = -\chi(F)$, $H_{-d\chi(F)}(map_*(h, X))$ is the multiplication by $(detH_1(h;\mathbb{Z}))^d$, $H_{-d\chi(F)}(map_*(\bar{f}, X))$ is also the multiplication by $(detH_1(h;\mathbb{Z}))^d$. Up to the isomorphisms

$$H_1(F, \partial_{in}F) \overset{\cong}{\to} H_1(F/\partial_{in}F) \underset{\cong}{\overset{H_1(g)^{-1}}{\to}} H_1(\vee_{-\chi(F)} S^1),$$

$H_1(f;\mathbb{Z}) : H_1(F, \partial_{in}F) \to H_1(F, \partial_{in}F)$ coincides with $H_1(h;\mathbb{Z}) : H_1(\vee_{-\chi(F)} S^1) \to H_1(\vee_{-\chi(F)} S^1)$. In particular, their determinants, $detH_1(f;\mathbb{Z})$ and $detH_1(h;\mathbb{Z})$, are equals. Since f is orientation preserving, by Proposition 19, $detH_1(f;\mathbb{Z})$ is $+1$. So finally,

$$H_{-d\chi(F)}(map_*(\bar{f}, X)) : H_{-d\chi(F)}(map_*(F/\partial_{in}F, X)) \to H_{-d\chi(F)}(map_*(F/\partial_{in}F, X))$$

is the identity morphism. □

5. Prop structure

In this section we prove that the action of evaluation products is propic, we thus have to prove that the evaluation products are compatible with the action of the symmetric group on the boundary components, with the gluing of surfaces along their boundaries and with the disjoint union of surfaces.

Proposition 24. *If F_1 and F_2 are two equivalent smooth cobordisms (Definition 7) then the evaluation products $\mu(F_1)$ and $\mu(F_2)$ coincides.*

Notation 25. Let S_n denote the disjoint union of n circles, $\coprod_{i=1}^n S^1$.

Proposition 26. *For any cobordism F, the restriction of the evaluation product*

$$\mu(F) : H_*(BD(F)) \otimes H_*(\mathcal{L}X)^{\otimes p} \overset{H_*(\rho_{out})\circ\rho_{in}!}{\to} H_*(\mathcal{L}X)^{\otimes q}$$

to $H_0(BD(F))$ *coincides with the operation induced by* F

$$H_*(\mathcal{L}X)^{\otimes p} \overset{in!}{\to} H_*(map(F,X)) \overset{H_*(out)}{\to} H_*(\mathcal{L}X)^{\otimes q}.$$

Proof. Consider the following commutative diagram where all the squares are pull-backs and the horizontal maps, fibrations.

$$\begin{tikzcd}
BD(F)\times map(S_p,X) & (map(F,X))_{hD(F)} \arrow[l, "\rho_{in}"'] \arrow[r] & BD(F)\times map(S_q,X) \\
ED(F)\times map(S_p,X) \arrow[u] \arrow[d, "proj_2"', "\approx"] & ED(F)\times map(F,X) \arrow[l, "Id\times in"] \arrow[r, "Id\times out"'] \arrow[u] \arrow[d, "\approx"', "proj_2"] & ED(F)\times map(S_q,X) \arrow[u] \arrow[d, "\approx"', "proj_2"] \\
map(S_p,X) & map(F,X) \arrow[l, "in"] \arrow[r, "out"'] & map(S_q,X)
\end{tikzcd}$$

Here $proj_2$ is the projection on the second factor. By definition, ρ_{out} fits into the commutative diagram

$$\begin{tikzcd}
map(F,X)_{hD(F)} \arrow[r] \arrow[d, "\rho_{out}"'] & BD(F)\times map(S_q,X) \arrow[dl, "proj_2"'] \\
map(S_q,X) & ED(F)\times map(S_q,X) \arrow[l, "\approx"', "proj_2"] \arrow[u]
\end{tikzcd}$$

By naturatility of integration along the fibers with respect to pull-backs, we obtain the proposition. $\square$

For $\varepsilon = 0$ or 1, let $i_\varepsilon : S_n \hookrightarrow S_n \times I$, $x \mapsto (x,\varepsilon)$, be the two canonical inclusions of S_n into the cylinder $S_n \times I$.

Proposition 27. *Let* $\phi : S_n \to S_n$ *be a diffeomorphism. Consider the cylinder* $S_n\times I$ *as the cobordism equipped with the in-boundary* $S_n \overset{i_0\circ\phi}{\hookrightarrow} S_n \times I$ *and the out-boundary* $S_n \overset{i_1}{\hookrightarrow} S_n \times I$*. Following* [42, 1.3.22]*, we denote this cobordism* C_ϕ*. The operation induced by* C_ϕ*:*

$$H_*(map(i_1,X))\circ map(i_0\circ\phi,X)! : H_*(\mathcal{L}X)^{\otimes n} \to H_*(map(C_\phi,X)) \to H_*(\mathcal{L}X)^{\otimes n}$$

coincides with $H_*(map(\phi,X))^{-1}$*.*

Proof. Since $map(-,X)$ preserves homotopies, the fibration $map(i_0,X)$ is a homotopy equivalence. So by naturality of integration along the fiber with respect to homotopy equivalences,

$$H_*(map(i_0,X))\circ map(i_0,X)! = id!\circ id = id.$$

But since $map(i_0,X)$ is homotopic to $map(i_1,X)$, $H_*(map(i_0,X))$ is equal to $H_*(map(i_1,X))$. Therefore we have proved the proposition in the case $\phi = id$. The general case follows since $map(\phi,X)! = H_*(map(\phi,X))^{-1}$ and $map(i_0\circ\phi,X)! = map(i_0,X)!\circ map(\phi,X)!$. $\square$

Let $id : S_n \to S_n$ be the identity diffeomorphism. Denote by $id_n \in H_0(BD(C_{id}))$ the identity morphisms of the prop. Let $\phi : S_m \coprod S_n \overset{\cong}{\to} S_n \coprod S_m$ be the twist diffeomorphism. Denote by $\tau_{m,n} \in H_0(BD(C_\phi))$ the symmetry isomorphism of the prop. From the previous two propositions, we immediately obtain

Corollary 28. *i) (identity) The evaluation product for the cylinder satisfies*
$\mu(C_{id})(id_n \otimes a_1 \otimes \cdots \otimes a_n) = a_1 \otimes \cdots \otimes a_n$ *for* $a_1, \ldots, a_n \in H_*(\mathcal{L}X)$.
ii) (symmetry) The evaluation product for C_ϕ *satisfies*
$\mu(C_\phi)(\tau_{m,n} \otimes v \otimes w) = (-1)^{|v||w|} w \otimes v$ *for* $v \in H_*(\mathcal{L}X)^{\otimes m}$ *and* $w \in H_*(\mathcal{L}X)^{\otimes n}$.

The following Lemma is obvious.

Lemma 29. *Let D and D' two topological groups. Let X be a left D-space. Let X' be a left D'-space. Let Y be a topological space that we considered to be a trivial left D-space and a trivial left D'-space. Let $f : X \to Y$ be a D-equivariant map. Let $f' : X' \to Y$ be a D'-equivariant map. Consider the pull-back*

$$\begin{tikzcd} X'' \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ X' \arrow[r, "f'"'] & Y \end{tikzcd}$$

Then
i) X" is a sub $D \times D'$-space of $X \times X'$ and we have the pull-back

$$\begin{tikzcd} (X'')_{hD\times D'} \arrow[r] \arrow[d] & (X)_{hD} \arrow[d] \\ (X')_{hD'} \arrow[r] & Y. \end{tikzcd}$$

ii) In particular, we have a natural homeomorphism

$$(X \times X')_{hD\times D'} \cong (X)_{hD} \times (X')_{hD'}.$$

Proposition 30. *(Monoidal, i.e. disjoint union) Let F_{p+q} and $F'_{p'+q'}$ be two surfaces. For $a \otimes v \in H_*(BD_{p+q} \times \mathcal{L}X^{\times p})$ and $b \otimes w \in H_*(BD'_{p+q} \times \mathcal{L}X^{\times p'})$ The evaluation product satisfies*

$$\mu(F \amalg F')(a \otimes b \otimes v \otimes w) = (-1)^{|b||v| - d\chi_{F'}(|a|+|v|)} \mu(F)(a \otimes v) \otimes \mu(F')(b \otimes w).$$

Proof. Consider the two cobordisms

$$\amalg_{i=1}^{p} S^1 \overset{in}{\hookrightarrow} F \overset{out}{\hookleftarrow} \amalg_{i=1}^{q} S^1 \quad \text{and} \quad \amalg_{i=1}^{p'} S^1 \overset{in'}{\hookrightarrow} F' \overset{out'}{\hookleftarrow} \amalg_{i=1}^{q'} S^1.$$

By definition, the disjoint union of these two cobordisms is

$$\amalg_{i=1}^{p} S^1 \amalg \amalg_{i=1}^{p'} S^1 \overset{in\amalg in'}{\hookrightarrow} F \amalg F' \overset{out\amalg out'}{\hookleftarrow} \amalg_{i=1}^{q} S^1 \amalg \amalg_{i=1}^{q'} S^1.$$

Since we assume that F and F' have each at least one boundary component, we have the isomorphism of topological groups

$$Diff^+(F \amalg F'; \partial) \cong Diff^+(F; \partial) \times Diff^+(F'; \partial).$$

Denote by D the group $Diff^+(F; \partial)$ while $D' := Diff^+(F'; \partial)$. The homeomorphism $map(Y \coprod Z, X) \cong map(Y, Z) \times map(Z, X)$ is natural in Y and Z. Therefore using Lemma 29 ii), we have the commutative diagram

$$\begin{tikzcd} (map(F,X))_{hD} \times (map(F',X))_{hD'} \arrow[r, "\cong"] \arrow[d, "\rho_{in}\times\rho'_{in}"'] & (map(F \amalg F', X))_{hD\times D'} \arrow[d] \\ BD \times \mathcal{L}X^{\times p} \times BD' \times \mathcal{L}X^{\times p'} \arrow[r, "\cong"'] & ED \times ED'/D \times D' \times \mathcal{L}X^{\times p+p'} \end{tikzcd}$$

where the horizontal maps are homeomorphisms. The homotopy equivalence $ED \times ED' \overset{\simeq}{\to} E(D \times D')$ induces the commutative diagram

$$\begin{tikzcd}
ED \times ED' \times_{D\times D'} map(F \amalg F', X) \arrow[r, "\simeq"] \arrow[d] & E(D \times D') \times_{D\times D'} map(F \amalg F', X) \arrow[d, "\rho_{in}"] \\
ED \times ED'/D \times D' \times \mathcal{L}X^{\times p+p'} \arrow[r, "\simeq"'] & B(D \times D') \times \mathcal{L}X^{p+p'}
\end{tikzcd}$$

where the horizontal maps are homotopy equivalences. Using similar commutative diagrams for ρ_{out}, by naturality of integration along the fiber with respect to homotopy equivalences, we obtain the commutative diagram

$$\begin{tikzcd}
H_*(BD \times \mathcal{L}X^p \times BD' \times \mathcal{L}X^{p'}) \arrow[r, "\cong"] \arrow[d, "(\rho_{in}\times\rho'_{in})_!"'] & H_*(B(D \times D') \times \mathcal{L}X^{p+p'}) \arrow[d, "\rho_{in}"] \\
H_*((map(F, X))_{hD} \times (map(F', X))_{hD'}) \arrow[r, "\cong"] \arrow[d, "H_*(\rho_{out}\times\rho'_{out})"'] & H_*((map(F \amalg F', X))_{hD\times D'}) \arrow[d, "H_*(\rho_{out})"] \\
H_*(\mathcal{L}X^q \times \mathcal{L}X^{q'}) \arrow[r, "\cong"] & H_*(\mathcal{L}X^{q+q'})
\end{tikzcd}$$

where the horizontal maps are isomorphisms. □

We now show that the evaluation products defined in the preceding section are compatible with gluing. Let us pick two surfaces $F_{g,p+q}$ and $F'_{g',q+r}$, gluing these surfaces one gets a third surfaces $F"_{g",p+r}$ and an inclusion of groups $D_{g,p+q} \times D_{g',q+r} \hookrightarrow D_{g",p+r}$. Therefore the composite gl

$$BD_{g,p+q} \times BD_{g',q+r} \approx B(D_{g,p+q} \times D_{g',q+r}) \to BD_{g",p+r}$$

gives in homology the gluing morphism

$$gl : H_*(BD_{g,p+q}) \otimes H_*(BD_{g',q+r}) \to H_*(BD_{g",p+r}).$$

Proposition 31. *(Composition, i.e. gluing). For any $a_i \in H_*(\mathcal{L}X)$ and any $m_1 \in H_*(BD_{g,p+q})$ and $m_2 \in H_*(BD_{g',q+r})$ one has*

$$(gl(m_1 \otimes m_2))(a_1 \otimes \ldots \otimes a_p) = m_2(m_1(a_1 \otimes \ldots \otimes a_p)).$$

Proof. We have the two sequences of maps

$$BD_{g,p+q} \times BD_{g',q+r} \times \mathcal{L}X^{\times p} \to BD_{g",p+r} \times \mathcal{L}X^{\times p} \leftarrow \mathcal{M}_{g",p+r}(X) \to \mathcal{L}X^{\times r}$$

and

$$BD_{g,p+q}\times BD_{g',q+r}\times\mathcal{L}X^{\times p} \leftarrow BD_{g',q+r}\times\mathcal{M}_{g,p+q}(X) \to BD_{g',q+r}\times\mathcal{L}X^{\times q} \leftarrow \mathcal{M}_{g',q+r}(X) \to \mathcal{L}X^{\times r}.$$

The first induces the operation $gl(m_1 \otimes m_2)$ while the second induces $m_2 \circ m_1$. In order to compare these two operations we introduce the following intermediate moduli space :

$$\mathcal{M}_{g,g',p+q+r}(X) := (map(F"_{g",p+r}, X))_{hD_{g,p+q}\times D_{g',q+r}}.$$

Denote by $\rho_{in}(g, g', p + g + r) :=$

$$(in)_{hD_{g,p+q}\times D_{g',q+r}} : \mathcal{M}_{g,g',p+q+r}(X) \twoheadrightarrow BD_{g,p+q} \times BD_{g',q+r} \times \mathcal{L}X^{\times p}.$$

Denote by $proj_1$ the projection of the first factor. we have the commutative diagram :

$$\begin{tikzcd}
\mathcal{M}_{g,g',p+q+r}(X) \arrow[r, "glue"] \arrow[d, "\rho_{in}(g,g',p+g+r)"'] & \mathcal{M}_{g",p+r}(X) \arrow[d, "\rho_{in}"] \\
BD_{g,p+q} \times BD_{g',q+r} \times \mathcal{L}X^{\times p} \arrow[r, "gl\times Id"'] \arrow[d, "proj_1"'] & BD_{g",p+r} \times \mathcal{L}X^{\times p} \arrow[d, "proj_1"] \\
BD_{g,p+q} \times BD_{g',q+r} \arrow[r, "gl"'] & BD_{g",p+r}
\end{tikzcd}$$

The two composites of the vertical maps, $proj_1 \circ \rho_{in}(g, g', p + g + r)$ and $proj_1 \circ \rho_{in}$ are fiber bundles with the same fiber $map(F"_{g",p+r}, X)$. Therefore the total square is a pull-back. Since the lower square is obviously a pull-back, by associativity of pull-back, the upper square is also a pull-back. Thus, by property of the integration along the fibers in homology we have :

$$\mu_{g,g',p+r} \circ (gl \times Id)_* = (\rho_{out})_* \circ glue_* \circ (\rho_{in}(g, g', p + q + r))_!$$

thus $gl(m_1 \otimes m_2)(a_1 \otimes \dots a_p)$ is equal to

$$(\rho_{out})_* \circ glue_* \circ (\rho_{in}(g, g', p + q + r))_!(m_1 \otimes m_2 \otimes a_1 \dots \otimes a_p).$$

Now, we have to compare our second correspondence (corresponding to $m_2 \circ m_1$) to the correspondence

$$BD_{g,p+q} \times BD_{g',q+r} \times \mathcal{L}X^{\times p} \xleftarrow{\rho_{in}(g,g',p+q+r)} \mathcal{M}_{g,g',p+q+r}(X) \to \mathcal{L}X^{\times r}.$$

By definition, $F"_{g",p+r}$ is given by the push-out

$$\begin{tikzcd}
\coprod_{i=1}^{q} S^1 \arrow[r] \arrow[d] & F_{g,p+q} \arrow[d] \\
F'_{g,q+r} \arrow[r] & F"_{g",p+r}.
\end{tikzcd}$$

By applying $map(-, X)$, we obtain the pull-back

(32)
$$\begin{tikzcd}
map(F"_{g",p+r}, X) \arrow[r, "f"] \arrow[d] & map(F_{g,p+q}, X) \arrow[d, "map(out,X)"] \\
map(F'_{g,q+r}, X) \arrow[r, "map(in,X)"'] & \mathcal{L}X^{\times q}.
\end{tikzcd}$$

By applying the Lemma 29 i) to the previous pull-back, we obtain the pull-backs,

$$\begin{tikzcd}
\mathcal{M}_{g,g',p+q+r}(X) \arrow[r] \arrow[d] & BD_{g',q+r} \times \mathcal{M}_{g,p+q}(X) \arrow[r, "proj_2"'] \arrow[d, "BD_{g',q+r}\times\rho_{out}"] & \mathcal{M}_{g,p+q}(X) \arrow[d, "\rho_{out}"] \\
\mathcal{M}_{g',q+r}(X) \arrow[r, "\rho_{in}"'] & BD_{g',q+r} \times \mathcal{L}X^{\times q} \arrow[r, "proj_2"'] & \mathcal{L}X^{\times q}
\end{tikzcd}$$

Finally, we have obtain the diagram

$$\begin{tikzcd}
 & BD_{g',q+r}\times\mathcal{M}_{g,p+q}(X) \arrow[dl, "BD_{g',q+r}\times\rho_{in}"'] \arrow[r, "BD_{g',q+r}\times\rho_{out}"] & BD_{g',q+r}\times\mathcal{L}X^{\times q} \\
BD_{g,p+q}\times BD_{g',q+r}\times\mathcal{L}X^{\times p} \arrow[d, "gl\times\mathcal{L}X^{\times p}"'] & \mathcal{M}_{g,g',p+q+r}(X) \arrow[l, "\rho_{in}(g,g',p+g+r)"] \arrow[r] \arrow[u] \arrow[d, "glue"] & \mathcal{M}_{g',q+r}(X) \arrow[u, "\rho_{in}"'] \arrow[d, "\rho_{out}"] \\
BD_{g",p+r}\times\mathcal{L}X^{\times p} & \mathcal{M}_{g",p+r}(X) \arrow[l, "\rho_{in}"] \arrow[r, "\rho_{out}"'] & \mathcal{L}X^{\times r}
\end{tikzcd}$$

where:

-the lower left square and the upper right square are pull-backs and

-the lower right square and the upper left triangle commutte.

The comparison follows from the Composition lemma, □

6. Results for connected topological groups

6.1. **Main Theorem.** In Section 4, we have defined for every oriented cobordism F whose path-conponents have at least one in-boundary component and at least one out-going boundary component, a linear map of degre $-d\chi(F)$,

$$\mu(F_{g,p+q}) : H_*(BD_{g,p+q})\otimes H_*(\mathcal{L}X)^{\otimes p}\to H_*(\mathcal{L}X)^{\otimes q}.$$

In Section 5, we have shown that these $\mu(F)$ define an action of the corresponding prop $H_*(BD)$ on $H_*(\mathcal{L}X)$. Therefore, by Proposition 11, we have our main Theorem:

Theorem 33. (Main Theorem) *Let X be a simply connected topological space such that the singular homology of its based loop space with coefficient in a field, $H_*(\Omega X,\mathbb{F})$, is finite dimensional. Then the singular homology of $\mathcal{L}X$ taken with coefficients in a field, $H_*(\mathcal{L}X;\mathbb{F})$, is a non-unital non-counital homological conformal field theory. (See Section 3.7 for the definition)*

6.2. **TQFT structure on $H_*(\mathcal{L}X)$.** In Sections 3.5 or 3.6, we recalled that there is an isomorphim of discrete props $\pi_0(\mathfrak{M})\cong\pi_0(BD)\cong sk(2-Cob)$ between the path-components of the two topological props $\mathfrak{M}$, BD and the skeleton of the category of oriented 2-dimensional cobordisms. Therefore, we have an inclusion of linear props

$$\mathbb{F}[sk(2-Cob)]\cong H_0(BD)\hookrightarrow H_*(BD).$$

A homological conformal field theory is an algebra over the linear prop $H_*(BD)$ (Section 3.7). A 2-dimensional topological quantum field theory is an algebra over the discrete prop $sk(2-Cob)$ (section 3.3). So, any (non-counital non-unital) homological conformal field theory is in particular a (non-counital non-unital) topological quantum field theory. Notice that (non-counital non-unital) topological quantum field theory are exactly (non-counital non-unital) commutative Frobenius algebra [42, Theorem 3.3.2]. So finally Theorem 33 implies the following Corollary.

Corollary 34. *Let X be a simply connected topological space such that $H_*(\Omega X,\mathbb{F})$ is finite dimensional. Then $H_*(\mathcal{L}X;\mathbb{F})$ is a non-unital non-counital Frobenius algebra.*

As we have just explained, Corollary 34 is a direct consequence of Theorem 33. But let us notice that Corollary 34 is much simpler to prove than Theorem 33, because the classifying space $BD(F)$ is not needed to construct the operations associated to the topological quantum field theory structure (Proposition 26).

6.3. **Main coTheorem.** Let V be a homological conformal field theory which is positively graded and of finite type in each degree. Then transposition gives a morphism of props : $\mathcal{E}nd_V^{op} \to \mathcal{E}nd_{V^\vee}$ from the opposite prop of the endomorphims prop of V, to the endomorphisms prop of the linear dual of V. Since the category of complex cobordisms, i. e. Segal prop $\mathfrak{M}$ (Section 3.4), is isomorphic to its opposite category, the linear dual $V^\vee$ of V is again a homological conformal field theory. Therefore, from Theorem 33, we obtain:

Theorem 35. (Main coTheorem) *Let X be a simply-connected topological space such that its based loop space homology with coefficient in a field, $H_*(\Omega X, \mathbb{F})$, is finite dimensional. Then its free loop space cohomology taken with coefficients into a field, $H^*(\mathcal{L}X, \mathbb{F})$, is a non-unital non-counital homological field theory.*

6.4. **Batalin-Vilkovisky algebra structure.** In Section 3.4, we have recalled the Segal prop of Riemann surfaces $\mathfrak{M} = \mathfrak{M}(p,q)_{p,q\geq 0}$. Consider the sub-operad

$$\mathfrak{M}_0 = \mathfrak{M}_0(p)_{p\geq 0} \subset \mathfrak{M}(p,1)_{p\geq 0}$$

where we consider only path-connected Riemann surfaces of genus 0, with only one out-going boundary component. By [29, p. 282] (See also [49, Proposition 4.8]), there is a natural map of topological operads from the framed little 2-discs operad $f\mathcal{D}_2$ to $\mathfrak{M}_0$ which is a homotopy equivalence. Alternatively, it is implicit in [70, Theorem 1.5.16] or [60, Theorem 7.3 and Proposition 7.4], that the framed little 2-discs operad $f\mathcal{D}_2(p)_{p\geq 0}$ is homotopy equivalent as operads to the operad $\{B\Gamma_{0,p+1}\}_{p\geq 0} \approx BDiff^+(F_{0,p+1}, \partial)_{p\geq 0}$.

Therefore any algebra over the linear prop $H_*(\mathcal{M}) \cong H_*(BD)$ is an algebra over the linear operad $H_*(f\mathcal{D}_2)$: any homological conformal field theory is in particular a Batalin-Vilkovisky algebra. Therefore, from Theorem 35, we deduce when the cohomology is taken with coefficient in a field:

Corollary 36. *Let $\Bbbk$ be any principal ideal domain. Let X be a simply connected space such*

- *For each $i \leq d$, $H_i(\Omega X; \Bbbk)$ is finite generated,*
- *$H_d(\Omega X; \Bbbk) \cong \Bbbk$, $H_{d-1}(\Omega X; \Bbbk)$ is $\Bbbk$-free and*
- *For $i > d$, $H_i(\Omega X; \Bbbk) = \{0\}$.*

Then the singular cohomology of $\mathcal{L}X$ with coefficients in $\Bbbk$, $H^*(\mathcal{L}X, \Bbbk)$, is an algebra over the operad $\oplus_{g\geq 0} H_*(BDiff^+(F_{g,p+1}))$, $p \geq 1$. In particular, the shifted cohomology $\mathbb{H}^*(\mathcal{L}X, \Bbbk) := H^{*+d}(\mathcal{L}X, \Bbbk)$ is a Batalin-Vilkovisky algebra (not necessarily with an unit).

Proof. We have already proved above the Corollary when $\Bbbk$ is a field. Over a principal ideal domain $\Bbbk$, we are going to indicate what are the differences. Note also that the proof of Corollary 41 will be similar.

Let $F_{g,p+q}$ be an oriented cobordism from $\coprod_{i=1}^p S^1$ to $\coprod_{i=1}^q S^1$. Using the universal coefficient theorem for cohomology and Kunneth theorem, we have that the cohomology of the fiber of ρ_{in} in degre $-d\chi(F)$

$$H^{-d\chi(F)} map_*(F/\partial_{in}F, X) \cong H^{-d\chi(F)}(\Omega X^{\times -\chi(F)}) \cong H^{-d}(\Omega X)^{\otimes -\chi(F)} \cong \Bbbk.$$

Therefore using integration along the fiber in cohomology, we can define the evaluation product

$$\mu(F)^* : H^l(\mathcal{L}X^{\times q}) \overset{H^l(\rho_{out})}{\to} H^l(\mathcal{M}_{q,p+p}(X)) \overset{\rho_{in}^!}{\to} H^{l+d\chi_F}(BD(F_{g,p+q}) \times \mathcal{L}X^p).$$

Now Section 5 implies that the evaluation products $\mu(F)^*$ are symmetric, compatible with gluing and disjoint union of cobordisms. Consider the composite, denoted $\mu(F)^\#$, of the tensor product of $\mu(F)^*$ with the identity morphism

$$H^l(\mathcal{L}X^{\times q}) \otimes H_m(BD(F)) \overset{\mu(F)^*\otimes Id}{\rightarrow} H^{l+d\chi_F}(BD(F) \times \mathcal{L}X^p) \otimes H_m(BD(F))$$

and of the Slant product [66, p. 254]

$$/ : H^{l+d\chi_F}(BD(F) \times \mathcal{L}X^p) \otimes H_m(BD(F)) \rightarrow H^{l+d\chi_F - m}(\mathcal{L}X^p).$$

Again, the evaluation products $\mu(F)^\#$ are symmetric, compatible with gluing and disjoint union. Therefore restricting to path-connected cobordisms $F_{g,1+q}$ with only $p = 1$ incoming boundary component, the composite

$$H^*(\mathcal{L}X)^{\otimes q} \otimes H_*(BD(F_{g,1+q})) \overset{Kunneth\otimes Id}{\rightarrow} H^*(\mathcal{L}X^{\times q}) \otimes H_*(BD(F_{g,1+q})) \overset{\mu(F)^\#}{\rightarrow} H^*(\mathcal{L}X)$$

defines an action of the opposite of the operad $H_*(BDiff^+(F_{g,1+q}, \partial))_{q\geq 1}$ on $H^*(\mathcal{L}X)$. But as recalled in Section 6.3, the topological operad $BDiff^+(F_{g,1+q}, \partial)_{q\geq 1}$ is isomorphic to the opposite of the operad $BDiff^+(F_{g,q+1}, \partial))_{q\geq 1}$. □

In the next section, we prove similar results for finite groups. But our results for finite groups are better than our results for connected topological groups.

-The main theorem for finite groups (Theorem 39) is in the homotopy category of spectra. In [5, (4.2)], there is a stable version of integration along the fiber for fibre bundles with smooth fibers. So Theorem 33 should also hold in the stable category.

-The structure for free loop space homology (respectively cohomology) for finite groups has a counit, (respectively an unit). When G is a connected compact Lie group, although we don't prove it, there is also an unit: the element $(EG \times_G \eta)^!(1) \in H^d(EG \times_G G^{ad}) \cong H^d(LBG)$ considered in the proof of Theorem 54.

To summarize, we believe that all our results for finite groups should extend to connected compact Lie groups.

7. The case of finite groups

In this section,we consider a finite group G instead of a connected topological group. Using transfer instead of integration along the fibers, we prove that, for a finite group G, the free loop homology on a $K(G, 1)$, $H_*(\mathcal{L}(K(G, 1)))$, is a counital non-unital homological conformal field theory. In order to define the transfert maps $map(in, K(G, 1))_!$ and $\rho_{in!}$, we need to check the finiteness of all the fibers of ρ_{in}.

7.1. **finiteness of all the fibres is preserved by:**

-pull-back and homotopy equivalence: Consider a commutative diagram

$$\begin{tikzcd} E_1 \arrow[r, "g"] \arrow[d, "p_1"] & E_2 \arrow[d, "p_2"] \\ B_1 \arrow[r, "h"] & B_2 \end{tikzcd}$$

where p_1 and p_2 are two fibrations. Suppose that the diagram is a pull-back or that h and g are homotopy equivalence. Then for any $b_1 \in B_1$, the fibre of p_1 over b_1, $p_1^{-1}(b_1)$, is homotopic to the fiber of p_2 over $h(b_1)$, $p_2^{-1}(h(b_1))$.

-composition: Let $f : X \twoheadrightarrow Y$ and let $g : Y \twoheadrightarrow Z$ be two fibrations. Let $z \in Z$ be any element of Z. By pull-back of f, we obtain the fibration $f' : (g \circ f)^{-1}(z) \twoheadrightarrow g^{-1}(z)$. If the base space of f', $g^{-1}(z)$, and if all the fibres of f', $f'^{-1}(y)$, $y \in$

$g^{-1}(z)$, are homotopy equivalent to a finite CW-complex then the total space of f', $(g\circ f)^{-1}(z)$ is also homotopy equivalent to a finite CW-complex.

-Borel construction: Let $p : E \twoheadrightarrow B$ be a fibration and also a G-equivariant map. We have the pull-back of principal G-bundles

$$\begin{array}{ccc} EG\times E & \xrightarrow{EG\times p} & EG\times B \\ \downarrow & & \downarrow \\ E_{hG} & \xrightarrow[p_{hG}]{} & B_{hG} \end{array}$$

Therefore the fiber of p_{hG} over the class $[x, b]$, $x \in EG$, $b \in B$, is homeomorphic to the fibre of p over b, $p^{-1}(b)$.

7.2. **Finiteness of the fibres of** $map(in, K(G, 1)$.

Proposition 37. *Let X be a path-connected space. Let $F_{g,p+q}$ be a path-connected oriented cobordism from $\coprod_{i=1}^{p} S^1$ to $\coprod_{i=1}^{q} S^1$. If $p \geq 1$ and $q \geq 1$ then all the fibers of the fibration obtained by restriction to the in-boundary components*

$$map(in, X) : map(F_{g,p+q}, X) \twoheadrightarrow \mathcal{L}X^{\times p}$$

are homotopy equivalent to the product, $\Omega X^{-\chi(F)}$.

Proof. The proof of this Proposition follows the same pattern as the proof of Proposition 15: Consider the commutative diagram (16). As $X^{\#v(c)}$ is path connected, all the fibres of the fibration

$$\prod_{v\in\sigma(c)} map(v, X) \twoheadrightarrow \prod_{v\in\sigma(c)} X^{\#\mu(v)} = X^{\#v(c)}$$

are homotopy equivalent. By pull-back and then by homotopy equivalence (Section (7.1)), all the fibres of the fibration $map(in, X) : map(F_{g,p+q}, X) \twoheadrightarrow (\mathcal{L}X)^{\times p}$ are homotopy equivalent. Let $x_0 \in X$. Denote by $\bar{x_0}$ the constant map from $\partial_{in}F$ to X. By Proposition 14, the fibre of $map(in, X)$ over $\bar{x_0}$, the pointed mapping space $map_*((F/\partial_{in}F, \partial_{in}F)(X, x_0))$ is homotopy equivalent to the product of pointed loop spaces, $\Omega X^{-\chi(F)}$. $\square$

Proposition 38. *Let G be a finite group. Let X be a $K(G, 1)$. Let $F_{g,p+q}$ be a path-connected oriented cobordism from $\coprod_{i=1}^{p} S^1$ to $\coprod_{i=1}^{q} S^1$.*

1) If $p \geq 1$ and $q \geq 0$ then all the fibers of the fibration obtained by restriction to the in-boundary components

$$map(in, X) : map(F_{g,p+q}, X) \twoheadrightarrow \mathcal{L}X^{\times p}$$

are homotopy equivalent to a discrete finite set.

2) Let R be a ring spectrum. If X is R-small then for any $q \geq 0$, the mapping space $map(F_{g,0+q}, X)$ is R-small

Proof. 1) The case $p \geq 1$ and $q \geq 1$ follows from the Proposition 37, since ΩX is homotopy equivalent to G.

Case $F_{0,1+0}$: Existence of the counit (to be compared with [16, Section 3] for the free loop space homology $H_*(\mathcal{L}M)$ of a manifold). Consider the disk D^2 as an oriented cobordism $F_{0,1+0}$ witn one incoming boundary and zero outgoing boundary. Let $l \in \mathcal{L}X$ be a free loop. The loop l is homotopic to a constant loop if and only if

l can be extend over D^2, the cone of S^1. This means exactly that l belongs to the image of

$$map(in, X) : map(D^2, X) \twoheadrightarrow \mathcal{L}X.$$

Let $x_0 \in X$. Let $\bar{x_0}$ be the constant free loop equal to x_0. The fibre of $map(in, X)$ over $\bar{x_0}$ is the double loop space $map_*((D^2/S^1, S^1)(X, x_0)) = \Omega^2(X, x_0)$. The double loop space $\Omega^2(X, x_0)$ is homotopy equivalent to ΩG, which is a point since G is discrete. Therefore all the fibres of $map(in, X) : map(D^2, X) \twoheadrightarrow \mathcal{L}X$ are either empty or contractile.

Case $p \geq 1$ and $q = 0$. By removing the interior of an embedded disk D^2 from the oriented cobordism $F_{g,p+0}$, we obtain an oriented cobordism $F_{g,p+1}$. According to the case $p \geq 1$ and $q \geq 1$, the operation associated to $F_{g,p+1}$

$$H_*(\mathcal{L}X)^{\times p} \overset{map(in,X)_!}{\longrightarrow} H_*(map(F_{g,p+1}, X)) \overset{map(out,X)}{\longrightarrow} H_*(\mathcal{L}X)$$

is defined. The operation associated to $D^2 = F_{0,1+0}$, namely the counit,

$$H_*(\mathcal{L}X) \overset{map(in,X)_!}{\longrightarrow} H_*(map(D^2, X)) \overset{map(out,X)}{\longrightarrow} H_*(map(\emptyset, X)) = H_*(point) = \mathbb{F}$$

is also defined. Therefore by composition, the operation associated to $F_{g,p+0}$

$$H_*(\mathcal{L}X)^{\times p} \overset{map(in,X)_!}{\longrightarrow} H_*(map(F_{g,p+0}, X)) \overset{map(out,X)}{\longrightarrow} H_*(map(\emptyset, X)) = H_*(point) = \mathbb{F}$$

is going to be defined. More precisely, consider the commutative diagram

$$\begin{tikzcd}
 & & \mathcal{L}X^{\times p} \\
 & map(F_{g,p+0}, X) \arrow[ur, "map(in,X)"] \arrow[r] \arrow[dl, "map(out,X)"'] \arrow[d] & map(F_{g,p+1}, X) \arrow[u, "map(in,X)"'] \arrow[d, "map(out,X)"] \\
\mathcal{L}X^{\times 0} & map(F_{0,1+0}, X) \arrow[l, "map(out,X)"] \arrow[r, "map(in,X)"'] & \mathcal{L}X^{\times 1}
\end{tikzcd}$$

where the square is the pull-back (32). Since all the fibres of $map(in, X) : map(F_{0,1+0}, X) \twoheadrightarrow \mathcal{L}X^{\times 1}$ are homotopy equivalent a finite discrete set, by pull-back (Section 7.1), all the fibres of the fibration $map(F_{g,p+0}, X) \twoheadrightarrow map(F_{g,p+1}, X)$ are also homotopy equivalent to a finite discrete set. Therefore, by composition (Section 7.1), all the fibres of $map(in, X) : map(F_{g,p+0}, X) \twoheadrightarrow \mathcal{L}X^{\times p}$ are homotopy equivalent to a finite discrete set.

Note that the same proof (Proof of Lemma 2) shows that the operation associated to $F_{g,p+0}$ is the composite of the operations associated to $F_{g,p+1}$ and $F_{g,1+0}$ (This is Proposition 31 without the $BDiff^+(F, \partial)$, compare also with Lemma 8 and Corollary 9 in [16]).

2) Consider this time, the disk D^2 as an oriented cobordism $F_{0,0+1}$ with zero incoming boundary and one outgoing boundary. We suppose that X is R-small. Since the fibre of the fibration $map(in, X) : map(D^2, X) \twoheadrightarrow map(\emptyset, X) = point$ is homotopy equivalent to X, the operation associated to $D^2 = F_{0,0+1}$, namely the unit,

$$R = R \wedge \Sigma^\infty map(\emptyset, X)_+ \overset{\tau_{map(in,X)}}{\longrightarrow} R \wedge \Sigma^\infty map(D^2, X)_+ \overset{R \wedge \Sigma^\infty map(out,X)_+}{\longrightarrow} R \wedge \Sigma^\infty \mathcal{L}X_+$$

is defined using Dwyer's transfert. By 1), the operation associated to $F_{g,1+q}$ is defined for all $q \geq 0$. Therefore, by composition (same arguments as in the proof of

1), in the case $p \geq 1$ and $q = 0$), the operation associated to $F_{g,0+q}$,

$$R \overset{\tau_{map(in,X)}}{\longrightarrow} R \wedge \Sigma^\infty map(F_{g,0+q}, X)_+ \overset{R\wedge\Sigma^\infty map(out,X)_+}{\longrightarrow} R \wedge (\Sigma^\infty \mathcal{L}X_+)^{\wedge q}$$

is also defined using Dwyer's transfert. That is the fibre of the fibration

$$map(in, X) : map(F_{g,0+q}, X) \twoheadrightarrow map(\emptyset, X) = point$$

is R-small. $\square$

7.3. **The results for finite groups.** Let $F_{g,p+q}$ be an oriented cobordism from $\amalg_{i=1}^p S^1$ to $\amalg_{i=1}^q S^1$. Let $Diff^+(F; \partial)$ be the group of orientation preserving diffeomorphisms that fix the boundaries pointwise.

Let G be a finite discrete group. Let X be a $K(G, 1)$. Suppose that every path component of $F_{g,p+q}$ has at least one in-boundary component. By part 1) of Proposition 38, all the fibers of the fibration $map(in; X) : map(F_{g,p+q}, X) \twoheadrightarrow \mathcal{L}X^{\times p}$ are homotopy equivalent to a finite CW-complex. Therefore by Section 7.1, all the fibres of the fibration obtained by Borel construction $(-)_{hDiff^+(F;\partial)}$

$$\rho_{in} := map(in, X)_{hDiff^+(F;\partial)} : map(F_{g,p+q}, X)_{hDiff^+(F;\partial)} \twoheadrightarrow BDiff^+(F; \partial) \times \mathcal{L}X^{\times p}$$

are homotopy equivalent to a finite CW-complex.

Therefore we can define the *evaluation product* associated to $F_{g,p+q}$.

$$\mu(F) : \Sigma^\infty BDiff^+(F; \partial)_+ \wedge (\Sigma^\infty \mathcal{L}X_+)^{\wedge p} \to (\Sigma^\infty \mathcal{L}X_+)^{\wedge q}$$

by the composite of the transfert map of ρ_{in},

$$\tau_{\rho_{in}} : \Sigma^\infty BDiff^+(F; \partial)_+ \wedge (\Sigma^\infty \mathcal{L}X_+)^{\wedge p} \to \Sigma^\infty map(F_{g,p+q}, X)_{hDiff^+(F;\partial)+}$$

and of

$$\Sigma^\infty \rho_{out+} : \Sigma^\infty map(F_{g,p+q}, X)_{hDiff^+(F;\partial)+} \to (\Sigma^\infty \mathcal{L}X_+)^{\wedge q}.$$

Now the same arguments as in Section 5 give a stable version of the main theorem:

Theorem 39. *(Stable version) Let G be a finite group. Let X be a $K(G, 1)$. Then the suspension spectrum $\Sigma^\infty \mathcal{L}X_+$ is an algebra over the stable prop of surfaces $\Sigma^\infty(BD)_+$ in the stable homotopy category (The topological prop BD is defined in Proposition 11 except that we consider here the cobordisms whose path components all have at least one in-boundary components).*

Corollary 40. *(Stable Frobenius algebra) Let G be a finite group. Let X be a $K(G, 1)$. Then the suspension spectrum $\Sigma^\infty \mathcal{L}X_+$ is a counital non-unital commutative Frobenius object (in the sense of* [42, 3.6.13]*) in the homotopy category of spectras. In particular the suspension spectrum $\Sigma^\infty \mathcal{L}X_+$ is a non-unital commutative associative ring spectrum.*

Proof. Consider the topological prop BD (or Segal prop $\mathfrak{M}$ since they are homotopy equivalent). Any map φ from a discrete set E to a topological space Y is uniquely determined up to homotopy by the composite $E \overset{\varphi}{\to} Y \twoheadrightarrow \pi_0(Y)$. Therefore the quotient map $BD \twoheadrightarrow \pi_0(BD)$ admits a section $\sigma : \pi_0(BD) \hookrightarrow BD$, which is a morphism of props up to homotopy since the quotient map $BD \twoheadrightarrow \pi_0(BD)$ is a morphism of props. Recall from Sections 3.5 or 3.6 that there is an isomorphim of discrete props $\pi_0(\mathfrak{M}) \cong \pi_0(BD) \cong sk(2 - Cob)$ between the path-components of the two topological props $\mathfrak{M}$, BD and the skeleton of the category of oriented 2-dimensional cobordisms. So we have a morphism of props $\Sigma^\infty sk(2 - Cob)_+ \hookrightarrow \Sigma^\infty(BD)_+$ in the stable homotopy category. Therefore by Theorem 39, $\Sigma^\infty \mathcal{L}X_+$

is an algebra over the stable prop $\Sigma^\infty sk(2-Cob)_+$. So by [42, Theorem 3.6.19], $\Sigma^\infty \mathcal{L}X_+$ is a commutative Frobenius object in the homotopy category of spectra. $\square$

Corollary 41. *(Batalin-Vilkovisky algebra) Let G be a finite group. Let X be a $K(G,1)$. Let h^* be a generalized cohomology theory coming from a commutative ring spectrum. Then $h^*(\mathcal{L}X)$, is an algebra over the operad $\oplus_{g\geq 0} h_*(BDiff^+(F_{g,p+1}))$, $p\geq 0$. In particular, the singular free loop space cohomology of X, with coefficients in any commutative ring $\Bbbk$, $H^*(\mathcal{L}X;\Bbbk)$, is an unital Batalin-Vilkovisky algebra.*

Property 42. In the homotopy category of spectra, let Y be a coalgebra over an operad $O=O(n)_{n\geq 0}$. Let $\mu : O(n)\wedge Y \to Y^{\wedge n}$ be the evaluation product. Then the composition of

$$h^*(Y)^{\otimes n}\otimes h_*(O(n)) \to h^*(Y^{\wedge n})\otimes h_*(O(n)) \overset{h^*(\mu)\otimes h_*(O(n))}{\to} h^*(O(n)\wedge Y)\otimes h_*(O(n))$$

and of the slant product for generalized multiplicative cohomology [66, p. 270 iii)]

$$/ : h^*(O(n)\wedge Y)\otimes h_*(O(n)) \to h^*(Y)$$

makes $h^*(Y)$ into an algebra over the opposite of the operad $h_*(O)$.

Proof of Corollary 41. By Theorem 39, $\Sigma^\infty \mathcal{L}X_+$ is a coalgebra over the stable operad $\Sigma^\infty \vee_{g\geq 0} BDiff^+(F_{g,1+q})_+$, $q\geq 0$, or over the homotopy equivalent stable operad $\Sigma^\infty \mathfrak{M}(1,q)_+$, $q\geq 0$. By Property 42, $h^*(\mathcal{L}X)$, is an algebra over the operad $h_*(\mathfrak{M}(1,q))^{op}$, $q\geq 0$. But the topological operad $\mathfrak{M}(p,1)$, $p\geq 0$ is isomorphic to the opposite of the operad $\mathfrak{M}(1,q)$, $q\geq 0$. Therefore $h^*(\mathcal{L}X)$ is an algebra over the operad $h_*(\mathfrak{M}(p,1))$, $p\geq 0$.

By taking the zero genus part, as in Section 6.4, we obtain that $h^*(\mathcal{L}X)$, is an algebra over the operad $h_*(f\mathcal{D}_2)$. If h^* is any singular cohomology theory, $h^*(\mathcal{L}X)$, is an unital Batalin-Vilkovisky algebra. $\square$

From Theorem 39 and Corollary 40, we get immediately:

Theorem 43. *(Generalized homology version) Let G be a finite group. Let X be a $K(G,1)$. Let h_* be a generalized homology theory with a commutative product such that the Kunneth morphism $h_*(X)\otimes_{h_*(pt)} h_*(Y) \overset{\cong}{\to} h_*(X\times Y)$ is an isomorphism. Then*

i) The free loop space homology $h_*(\mathcal{L}X)$ is an algebra over the prop $h_*(BD)$ in the category of graded modules over the graded commutative algebra $h_*(pt)$.

ii) $h_*(\mathcal{L}X)$ is a non-unital counital Frobenius algebra in the category of $h_*(pt)$-modules.

In particular from i), we have:

Theorem 44. *(Singular homology version) Let G be a finite group. Let X be a $K(G,1)$. Then the singular free loop space homology of X, with coefficients in a field $\mathbb{F}$, $H_*(\mathcal{L}X;\mathbb{F})$, is a counital non-unital homological conformal field theory.(See Section 3.7 for the definition).*

Of course, there is also a singular cohomology version. In [59, p. 176, (B.7.3)], Ravenel explains that Morava K-theory and singular homology with field coefficients are essentially the only generalized homology theories where Kunneth is an isomorphism. For these generalized homology theory with Kunneth isomorphism, it turns out that, in many case, the Frobenius algebra $h_*(\mathcal{L}X)$ of Theorem 43 ii) has an unit:

Corollary 45. *(unit) Let G be a finite group. Let X be a $K(G,1)$. Then*

1) (Dijkgraaf-Witten) if char($\mathbb{F}$) does not divide card(G) then $H_*(\mathcal{L}BG;\mathbb{F}) = H_0(\mathcal{L}BG;\mathbb{F})$ is an unital and counital Frobenius algebra.

2) (Comparison with Strickland [64] below) if $K(n)$ is the even periodic Morava K-theory spectrum at an odd prime then $K(n)_*(\mathcal{L}BG)$ is an unital and counital Frobenius algebra in the category of $K(n)_*(pt)$-graded modules.

Proof. Let R be the Eilenberg-Mac Lane spectrum $H\mathbb{F}$ in case 1) and let R be $K(n)$ in case 2). By part 2) of Proposition 38, it suffices to show that X is R-small. As recalled in Section 2.1.3, this is the case if the homology $R_*(X)$ is finitely generated as $R_*(pt)$-module.

Case 1) Since the cardinal of G is invertible in $\mathbb{F}$, $H_*(X;\mathbb{F})$ is concentrated in degree 0 [10, dual of Chap III Corollary 10.2]. Therefore $H_*(X;\mathbb{F}) = H_0(X;\mathbb{F}) \cong \mathbb{F}$ is a finite dimension vector space over $\mathbb{F}$.

Case 2) If p is odd, the two-periodic Morava K-theory $K(n)$ is a commutative ring spectrum [64, p. 764]. By [58], $K(n)_*(X)$ is finitely generated as $K(n)_*(pt)$-modules. ☐

Remark 46. 1) When $\mathbb{F}$ is the field of complex numbers $\mathbb{C}$, we have not checked that our Frobenius algebra $H_0(\mathcal{L}BG;\mathbb{C})$ coincides with the Frobenius algebra of Dijkgraaf-Witten. But it should!

2) Let $\mathcal{G}$ be a finite groupoid. Let $B\mathcal{G}$ its classifying space. In [64, Theorem 8.7], Strickland showed that the suspension spectrum of $B\mathcal{G}$ localized with respect to $K(n)$ is an unital and counital Frobenius object. Roughly, the comultiplication is the diagonal map $B\mathcal{G} \to B\mathcal{G} \otimes B\mathcal{G}$. The counit is the projection map $B\mathcal{G} \to *$. On the contrary, the multiplication is given by the transfer map of the evaluation fibration $(ev_0, ev_1) : B\mathcal{G}^{[0,1]} \twoheadrightarrow B\mathcal{G} \otimes B\mathcal{G}$. The unit is the transfert of the projection map $B\mathcal{G} \to *$. As pointed by Strickland [64, p. 733], this Frobenius structure has "striking formal similarities" with the case of manifolds (See 3) of example 49). In particular, $K(n)_*(B\mathcal{G})$ is an unital and counital Frobenius algebra in the category of $K(n)_*(pt)$-modules.

Let G be a finite group. The inertia groupoid ΛG of G ([64, Definition 8.8] or [3, Definition 2.49]) is a finite groupoid whose classifying space $B\Lambda G$ is homotopy equivalent to the free loop space on BG, $\mathcal{L}BG$. Therefore applying Strickland results, we obtain that $K(n)_*(\mathcal{L}BG)$ is an unital and counital Frobenius algebra like in part 2) of our Corollary 45. We have not checked that Strickland Frobenius algebra coincides with ours. But Strickland definitions of the comultiplication, of the counit, of the multiplication and of the unit are very different from the definitions using cobordism given in this paper.

Remark 47. By Proposition 37, Theorem 39 can be extended to path-connected spaces X such that ΩX is (stably) equivalent to a finite CW-complex. In this case, we are only able to get a non-counital non-unital homological conformal field theory. Although we have not prove it, we believe that this structure is trivial except when $H_*(\Omega X;\mathbb{F})$ is concentrated in degre 0.

8. Frobenius algebras and symmetric Frobenius algebras

In this section, we recall the notion of symmetric Frobenius and Frobenius algebras and prove that the homology of a connected compact Lie group is a symmetric Frobenius algebra.

8.1. **Frobenius algebras.** Frobenius algebras arise in the representation theory of algebras.
A Frobenius algebra is a finite dimensional unitary associative algebra A over a field R, equipped with a bilinear form called the Frobenius form

$$< -, - >: A \otimes A \to \mathbb{F}$$

that satisfies the Frobenius identity

$$< a, bc >=< ab, c >$$

and is non-degenerate.
They can also be characterized by the existence of an isomorphism $\lambda_L : A \cong A^\vee$ of left-A-modules. In fact the existence of such an isomorphism implies the existence of a coassociative counital coproduct

$$\delta : A \to A \otimes A$$

which is a morphism of A-bimodules [1, Thm 2.1]. Here the A-bimodule structure on $A \otimes A$ is the *outer* bimodule structure given by

$$a\dot{(}b \otimes b')\dot{c} := ab \otimes b'c$$

for a, b, b' and $c \in A$.
When $\lambda_L : A \cong A^\vee$ is an isomorphism of A-bimodule, the algebra A is called symmetric Frobenius, this is equivalent to requiring that the Frobenius form is symmetric $< a, b >=< b, a >$. Let us notice that a commutative Frobenius algebra is always a symmetric Frobenius algebra.

Example 48. 1) A classical example is given by algebras of matrices $M_n(\mathbb{F})$ where

$$< A, B >= tr(A.B).$$

2) Let G be a finite group then its group algebra $\mathbb{F}[G]$ is a non commutative symmetric Frobenius algebra.By definition, the group ring $\mathbb{F}[G]$ admits the set $\{g \in G\}$ as a basis. Denote by δ_g the dual basis in $\mathbb{F}[G]^\vee$. The linear isomorphism $\lambda_L : \mathbb{F}[G] \to \mathbb{F}[G]^\vee$, sending g to $\delta_{g^{-1}}$ is an isomorphism of $\mathbb{F}[G]$-bimodules.
3) Let M^d be a compact oriented closed manifold of dimension d then the singular homology $H_{*+d}(M^d, \mathbb{F})$ is a commutative Frobenius algebra of (lower) degree $+d$. The product is the intersection product, the coproduct is induced by the diagonal $\Delta : M^d \to M^d \times M^d$. The counit is induced by the projection map $M \to *$. The unit is the orientation class $[M] \in H_d(M)$.

8.2. **Hopf algebras.** Let H be a finite dimensional Hopf algebra over a field $\mathbb{F}$. A left (respectively right) *integral* in H is an element l of H such that $\forall h \in H$, $h \times l = \varepsilon(h)l$ (respectively $l \times h = \varepsilon(h)l$). A Hopf algebra H is *unimodular* if there exists a non-zero element $l \in H$ which is both a left and a right integral in H.

Example 49. If G is a finite group, $\sum_{g \in G} g$ is both a left and right integral in the group algebra $\mathbb{F}[G]$.

The set $\int$ of left (respectively right) integrals in the dual Hopf algebra $H^\vee$ is a $\mathbb{F}$-vector space of dimension 1 [65, Corollary 5.1.6 2)]. Let λ be any non-zero left (respectively right) integral in $H^\vee$. The morphism of left (respectively right) H-modules, $H \overset{\cong}{\to} H^\vee$ sending 1 to λ is an isomorphism [65, Proof of Corollary 5.1.6 2)]. So a finite dimensional Hopf algebra is always a Frobenius algebra, but not always a symmetric Frobenius algebra as the following theorem shows.

Theorem 50. *(Due to* [57]*. Other proofs are given in* [23] *and* [45, p. 487 Proposition]*. See also* [39]*.) A Hopf algebra H is a symmetric Frobenius algebra if and only if H is unimodular and its antipode S satisfies S^2 is an inner automorphism of H.*

Assume that H is unimodular and that S^2 is an inner automorphism of H. Let u be an invertible element $u \in H$ such that $\forall h \in H$, $S^2(h) = uhu^{-1}$. Let λ be any non-zero left integral in $H^\vee$. Then $\beta(h,k) := \lambda(hku)$ is a non-degenerate symmetric bilinear form [45, p. 487 proof of Proposition].

Example 51. Let G be a finite group. Since $S^2 = Id$ and since δ_1 is a left integral for $\mathbb{F}[G]^\vee$, we recover that the the linear isomorphism $\mathbb{F}[G] \to \mathbb{F}[G]^\vee$, sending g to $\delta_1(-g) = \delta_{g^{-1}}$ is an isomorphism of $\mathbb{F}[G]$-bimodules.

All the previous results extend to graded Hopf algebras. We need the following.

Proposition 52. *Let H be a cocommutative (lower) graded Hopf algebra such that*
i) $H_0 \cong \mathbb{F}$ (H is connected),
ii) H is concentrated in degrees between 0 and d and
iii) $H_d \neq 0$.
Then there exists an isomorphism $H \overset{\cong}{\to} H^\vee$ of H-bimodules (necessarily of (lower) degree $-d$:$H_p \overset{\cong}{\to} (H_{d-p})^\vee$), i.e. H is a symmetric Frobenius algebra of (lower) degree $-d$.

Proof. Since H is cocommutative, $S^2 = Id$. For degree reasons, any element $l \in H_d$ is both a left and a right integral. Since $H_d \neq 0$, H is unimodular. Therefore, by Theorem 50, H is a symmetric Frobenius algebra. □

Notice that an ungraded cocommutative Hopf algebra can be non unimodular [45, p. 487-8, Remark and Examples (1) and (4)]. Therefore the previous Proposition is false without condition i).

8.3. **The case of compact Lie groups.** Let us come to our motivational example. Let us take a connected compact Lie group G of dimension d. Let m denotes the product and Inv the inverse map of G. As a manifold one knows that $H_{*+d}(G,\mathbb{F})$ together with the intersection product is a commutative Frobenius algebra of lower degree $+d$. Moreover its homology together with the coproduct Δ_* and the Pontryagin product m_* is a finite dimensional connected cocommutative Hopf algebra, the antipode map S is given by $S = Inv_*$. Therefore using the above Proposition, $H_*(G)$ together with the Pontryagin product is a symmetric Frobenius algebra of (lower) degree $-d$. Denote by

$$\Theta : H_p(G) \overset{\cong}{\to} (H_{d-p}(G))^\vee \cong H^{d-p}(G)$$

an isomorphism of $H_*(G)$-bimodules of (lower) degree $-d$. Let $\eta_!$ be the Poincaré dual of the canonical inclusion $\eta : \{1\} \subset G$. Since $\eta_!$ is a non-zero element of $H_d(G)^\vee$, there exists a non-zero scalar $\alpha \in \mathbb{F}$ such that $\eta_! = \alpha.\Theta(1)$. Therefore, we have obtained

Theorem 53. *The singular homology of a connected compact Lie group G taken with coefficients in a field and equipped with the bilinear pairing*

$$< a, b >:= \eta_!(m_*(a \otimes b))$$

is a symmetric Frobenius algebra.

Second Proof. Let us give a more topological proof. The Frobenius relation is automatically satisfied. This bilinear form is closely related to the intersection product. In fact one can consider the following pull-back diagram :

$$\begin{tikzcd} G \arrow[r, "p"] \arrow[d, "\Delta"] & \{e\} \arrow[d, "\eta"] \\ G \times G \arrow[r, "m'"'] & G \end{tikzcd}$$

where $m'(h,g) = h^{-1}g$. Using properties of Poincaré duality with respect to Pull-back diagrams and denoting by $\bullet$ the intersection product (Poincaré dual of Δ) one gets :

$$p_*(a \bullet b) = \eta_! \circ m_*(S(a) \otimes b).$$

Using $m''(h,g) = hg^{-1}$, we find the relation

$$p_*(a \bullet b) = \eta_! \circ m_*(a \otimes S(b)).$$

Therefore,

$$< a, b >:= \eta_!(m_*(a \otimes b)) = p_*(S^{-1}(a) \bullet b) = p_*(a \bullet S^{-1}(b)).$$

Since the intersection product is commutative, the pairing $< a, b >$ is symmetric. We recall that $p_*(a \bullet b)$ is the Frobenius form associated to the Frobenius structure on $H_*(G, \mathbb{F})$ given by Poincaré duality. Therefore since S^{-1} is an isomorphism, the pairing $< a, b >$ is non-degenerate. □

9. Hochschild cohomology

Let G be a finite group. The group ring $\mathbb{F}[G]$ is equipped with an isomorphism

$$\lambda_L : \mathbb{F}[G] \overset{\cong}{\to} \mathbb{F}[G]^\vee$$

of $\mathbb{F}[G]$-bimodules and is therefore a symmetric Frobenius algebra (Example 48 2)). So we have the induced isomorphism in Hochschild cohomology

$$HH^*(\mathbb{F}[G]; \lambda_L) : HH^*(\mathbb{F}[G]; \mathbb{F}[G]) \overset{\cong}{\to} HH^*(\mathbb{F}[G]; \mathbb{F}[G]^\vee).$$

Our inspirational theorem in section 2 says that the Gerstenhaber algebra $HH^*(\mathbb{F}[G]; \mathbb{F}[G])$ is a Batalin-Vilkovisky algebra. Here the Δ operator is the Connes coboundary map $H(B^\vee)$ on $HH^*(\mathbb{F}[G]; \mathbb{F}[G]^\vee)$. In this section, we extends our inspirational theorem for finite groups to connected compact Lie groups:

Theorem 54. *Let G be a connected compact Lie group of dimension d. Denote by $S_*(G)$ the algebra of singular chains of G. Consider Connes coboundary map $H(B^\vee)$ on the Hochschild cohomology of $S_*(G)$ with coefficients in its dual, $HH^*(S_*(G); S^*(G))$. then there is an isomorphism of graded vector spaces of upper degree d*

$$\mathbb{D}^{-1} : HH^p(S_*(G); S_*(G)) \overset{\cong}{\to} HH^{p+d}(S_*(G); S^*(G))$$

such that the Gerstenhaber algebra $HH^*(S_*(G); S_*(G))$ equipped with the operator $\Delta = \mathbb{D} \circ H(B^\vee) \circ \mathbb{D}^{-1}$ is a Batalin-Vilkovisky algebra.

The proof of this theorem relies on Propositions 10 and 11 of [53] and on the following three Lemmas. Denote by $\eta : \{1\} \hookrightarrow G$ the inclusion of the trivial group into G.

Lemma 55. *The morphism of left $H_*(G)$-modules*

$$H_p(G) \to H_{d-p}(G)^\vee, a \mapsto a.\eta_!$$

is an isomorphism.

This Lemma is a particular case of Theorem 53 in the previous section. But we prefer to give an independent and more simple proof of this Lemma. In fact in this section, we implicitly [53, Proof of Proposition 10] give two morphisms of $S_*(G)$-bimodules

$$S_*(G) \overset{\simeq}{\to} P \overset{\simeq}{\leftarrow} S_*(G)^\vee$$

which induce isomorphisms in homology. In particular, passing to homology, we obtain a third proof of Theorem 53.

Proof of Lemma 55. By [65, Proof of Corollary 5.1.6 2)], since $H_*(G)$ is a finite dimensional Hopf algebra, $H_*(G)$ together with the Pontryagin product is a Frobenius algebra: there exists an isomorphism of left $H_*(G)$-modules

$$\Theta : H_*(G) \overset{\cong}{\to} (H_*(G))^\vee.$$

Since $H_*(G)$ is concentrated in degrees between 0 and d and since $H_0(G)$ and $H_d(G)$ are two non trivial vector spaces, the isomorphism Θ must be of (lower) degree $-d$. Let $\eta_!$ be the Poincaré dual of the canonical inclusion $\eta : \{1\} \subset G$. Since $\eta_!$ is a non-zero element of $H_d(G)^\vee$, there exists a non-zero scalar $\alpha \in \mathbb{F}$ such that $\eta_! = \alpha.\Theta(1)$. Therefore the morphism of left $H_*(G)$-modules

$$\alpha\Theta : H_*(G) \overset{\cong}{\to} (H_*(G))^\vee$$

is an isomorphism. This is the desired isomorphism since $\alpha\Theta(1) = \eta_!$. $\square$

Let M and N be two oriented closed smooth manifolds of dimensions m and n. Let G be a connected compact Lie group acting smoothly on M and N. Let $f : M \to N$ be a smooth G-equivariant map. Then we have a *Gysin equivariant map* in homology

$$(EG \times_G f)_! : H_*(EG \times_G N) \to H_{*+m-n}(EG \times_G M)$$

and a Gysin equivariant map in cohomology [40, Theorem 6.1]

$$(EG \times_G f)^! : H^*(EG \times_G M) \to H^{*+n-m}(EG \times_G N).$$

Similarily to integration along the fiber, Gysin equivariant maps are natural with respect to pull-backs and products, since this is a general property of Gysin maps and equivariant Gysin maps can be viewed as particular cases of Gysin maps (See for example, Proposition 4.17 of [9]).

Lemma 56. *Let K be a connected compact Lie group. Suppose that $EG \times_G M$ and $EG \times_G N$ are two left K-spaces. Suppose also that*

$$EG \times_G f : EG \times_G M \to EG \times_G N$$

is K-equivariant. Then the Gysin equivariant map

$$(EG \times_G f)_! : H_*(EG \times_G N) \to H_{*+m-n}(EG \times_G M)$$

is a morphism of left $H_*(K)$-modules. In particular, if K is the circle,

$$\Delta \circ (EG \times_G f)_! = (EG \times_G f)_! \circ \Delta.$$

Proof. Consider the pull-back diagram:

$$\begin{tikzcd}
K \times EG \times_G M \arrow[r, "K\times EG\times_G f"] \arrow[d, "action"'] & K \times EG \times_G N \arrow[d, "action"] \\
EG \times_G M \arrow[r, "EG\times_G f"'] & EG \times_G N
\end{tikzcd}$$

where *action* are the actions of K on $EG \times_G M$ and $EG \times_G N$. By naturality of Gysin equivariant map with respect to this pull-back and to products,we obtain the commutative diagram

$$\begin{tikzcd}
H_*(K) \otimes H_*(EG \times_G M) \arrow[d] & H_*(K) \otimes H_*(EG \times_G N) \arrow[l, "H_*(K)\otimes(EG\times_G f)_!"'] \arrow[d] \\
H_*(K \times EG \times_G M) \arrow[d, "H_*(action)"'] & H_*(K \times EG \times_G N) \arrow[l, "(K\times EG\times_G f)_!"] \arrow[d, "H_*(action)"] \\
H_*(EG \times_G M) & H_*(EG \times_G N) \arrow[l, "(EG\times_G f)_!"']
\end{tikzcd}$$

□

Let us denote by G^{ad} the left G-space obtained by the conjugation action of G on itself. The inclusion $\eta : \{1\} \hookrightarrow G^{ad}$ is a G-equivariant embedding of dimension d. Therefore, we have a Gysin equivariant morphism

$$(EG \times_G \eta)^! : H^*(BG) \to H^{*+d}(EG \times_G G^{ad}).$$

Lemma 57. *The morphism*

$$H^d(E\eta \times_\eta G^{ad}) : H^d(EG \times_G G^{ad}) \to H^d(G)$$

maps $(EG \times_G \eta)^!(1)$ *to* $\eta_! \in H_d(G)^\vee$.

Proof. Consider the two commutative squares

$$\begin{tikzcd}
\{1\} \arrow[r] \arrow[d, "\eta"'] & BG \arrow[d, "EG\times_G\eta"] \\
G^{ad} \arrow[r, "E\eta\times_\eta G^{ad}"'] \arrow[d] & EG \times_G G^{ad} \arrow[d, "p"] \\
\{1\} \arrow[r] & BG
\end{tikzcd}$$

The lower square is a fiber product since G^{ad} is the fiber of the fiber bundle $p : EG \times_G G^{ad} \twoheadrightarrow BG$. The total square is a fiber product since $EG \times_G \eta$ is a section of p. Therefore the upper square is also a fiber product. By naturality of Gysin equivariant morphism with respect to fiber products, we obtain the commutative square

$$\begin{tikzcd}
\mathbb{F} \arrow[d, "\eta^!"'] & H^*(BG) \arrow[l] \arrow[d, "(EG\times_G\eta)^!"] \\
H^*(G^{ad}) & H^*(EG \times_G G^{ad}) \arrow[l, "H^*(E\eta\times_\eta G^{ad})"]
\end{tikzcd}$$

Therefore $H^*(E\eta \times_\eta G^{ad}) \circ (EG \times_G \eta)^!(1) = \eta^!(1)$. □

Proof of Theorem 54. Let M be a left G-space. Let $B(*;G;M)$ denote the simplicial Bar construction [51, p. 31]. Recall that its space of n-simplices $B(*;G;M)_n$ is the product $G^{\times n}\times M$. The realisation $|-|$ of this simplicial space, $|B(*;G;M)|$ is homeomorphic to the Borel Construction $EG\times_G M$ when we set $EG:=|B(*;G;G)|$ [51, p. 40]. Let ΓG be the cyclic Bar construction of G [44, 7.3.10]. The continuous application

$$\Phi:\Gamma G\rightarrow B(*;G;G^{ad})$$

$$(m,g_1,\cdots,g_n)\mapsto(g_1,\cdots,g_n,g_1\dots g_n m)$$

is an isomorphism of simplicial spaces. (In homological algebra, the same isomorphism [44, 7.4.2] proves that Hochschild homology and group homology are isomorphic). The simplicial space ΓG is in fact a cyclic space. Let us consider the structure of cyclic space on $B(*;G;G^{ad})$ such that Φ is an isomorphisn of cyclic spaces. Recall that $\eta:\{1\}\hookrightarrow G$ denote the inclusion of the trivial group into G. The composite

$$B(*;G;*)\overset{B(*;G;\eta)}{\rightarrow}B(*;G;G^{ad})\overset{\Phi^{-1}}{\rightarrow}\Gamma G$$

which maps the n-simplex $[g_1,\dots,g_n]$ of $B(*;G;*)$ to the n-simplex $(g_1\dots g_n)^{-1},g_1,\dots,g_n)$ of ΓG, is an injective morphism of cyclic spaces [44, 7.4.5]. Here the simplicial space $B(*;G;*)$ is equipped with the structure of cyclic space called twisted nerve and denoted $B(G,1)$ in [44, 7.3.3]. Since realisation is a functor from cyclic spaces to S^1-spaces, we obtain that

-the homeomorphism $|\Phi|:|\Gamma G|\overset{\cong}{\rightarrow}EG\times_G G^{ad}$ is S^1-equivariant and

-the inclusion $EG\times_G\eta:BG\hookrightarrow EG\times_G G^{ad}$ is also S^1-equivariant.

Therefore in homology we have $H_*(|\Phi|)\circ\Delta=\Delta\circ H_*(|\Phi|)$. By Lemma 56, we also have $(EG\times_G\eta)_!\circ\Delta=\Delta\circ(EG\times_G\eta)_!$. Finally by dualizing, in cohomology, we have

$$\Delta^\vee\circ H^*(|\Phi|)=H^*(|\Phi|)\circ\Delta^\vee\quad\text{and}\quad\Delta^\vee\circ(EG\times_G\eta)^!=(EG\times_G\eta)^!\circ\Delta^\vee. \tag{58}$$

Let $j:G\rightarrow\Gamma G$, $g\mapsto(g,1,\dots,1)$, the inclusion of the constant simplicial space G into ΓG. Consider the morphism of simplicial spaces

$$B(*;\eta;G^{ad}):G^{ad}=B(*;*;G^{ad})\rightarrow B(*;G;G^{ad})$$

$$g\mapsto(1,\dots,1,g).$$

Obviously $\Phi\circ j=B(*;\eta;G^{ad})$. Therefore we have the commutative diagram of topological spaces

$$\begin{tikzcd}
& G \arrow[dl, "E\eta\times_\eta G^{ad}"'] \arrow[dr, "|j|"] & \\
EG\times_G G^{ad} & & |\Gamma G| \arrow[ll, "|\Phi|"]
\end{tikzcd}$$

In [11, 32], Burghelea, Fiedorowicz and Goodwillie proved that there is an isomorphism of vector spaces between $H^*(\mathcal{L}BG)$ and $HH^*(S_*(G);S^*(G))$. More precisely, they give a S^1-equivariant weak homotopy equivalence $\gamma:|\Gamma G|\rightarrow\mathcal{L}BG$ [44, 7.3.11]. And they construct an isomorphism $BFG:H^*(|\Gamma G|)\rightarrow HH^*(S_*(G);S^*(G))$. Denote by $\eta_{S_*(G)}:\mathbb{F}\hookrightarrow S_*(G)$ the unit of the algebra $S_*(G)$. It is easy to check that

$$HH^*(\eta_{S_*(G)};S^*(G))\circ BFG=H^*(|j|).$$

Therefore we have the commutative diagram.

$$\begin{tikzcd}
 & H^*(G) & \\
H^*(EG\times_G G^{ad}) \arrow[ur, "H^*(E\eta\times_\eta G^{ad})"] \arrow[r, "H^*(|\Phi|)"'] & H^*(|\Gamma G|) \arrow[u, "H^*(|j|)"'] \arrow[r, "BFG"'] & HH^*(S_*(G);S^*(G)) \arrow[ul, "HH^*(\eta_{S_*(G)};S^*(G))"']
\end{tikzcd}$$

Note that this diagram is similar to the diagram in the proof of Theorems 20 and 21 in [53].

Let $m\in HH^d(S_*(G);S^*(G))$ be $BFG\circ H^*(|\Phi|)\circ(EG\times_G\eta)^!(1)$. By the above commutative diagram and Lemma 57,

$$HH^d(\eta_{S_*(G)};S^*(G))(m)=H^d(E\eta\times_\eta G^{ad})\circ(EG\times_G\eta)^!(1)=\eta_!.$$

Therefore, by Lemma 55, the morphism of left $H_*(G)$-modules

$$H_*(G)\to H_*(G)^\vee, a\mapsto a.HH^d(\eta_{S_*(G)};S^*(G))(m)$$

is an isomorphism. Therefore by Proposition 10 of [53], we obtain that the morphism of $HH^*(S_*(G);S_*(G))$-modules

$$\mathbb{D}^{-1}:HH^p(S_*(G);S_*(G))\overset{\cong}{\to}HH^{p+d}(S_*(G);S^*(G),$$
$$a\mapsto a.m$$

is an isomorphism.

The isomorphism of Burghelea, Fiedorowicz and Goodwillie

$$BFG:H^*(|\Gamma G|)\to HH^*(S_*(G);S^*(G))$$

is compatible with the action of the circle on $|\Gamma G|$ and the dual of Connes boundary map $H(B^\vee)$: this means that $BFG\circ\Delta^\vee=H(B^\vee)\circ BFG$. Since $\Delta^\vee$ is a derivation for the cup product, $\Delta^\vee(1)=0$. Therefore using (58),

$$\begin{aligned}H(B^\vee)(m)&=BFG\circ\Delta^\vee\circ H^*(|\Phi|)\circ(EG\times_G\eta)^!(1)\\&=BFG\circ H^*(|\Phi|)\circ(EG\times_G\eta)^!\circ\Delta^\vee(1)=0.\end{aligned}$$

So, by applying Proposition 11 of [53], we obtain the desired Batalin-Vilkovisky algebra structure. □

Remark 59. Denote by $s:BG\hookrightarrow\mathcal{L}BG$ the inclusion of the constants loops into $\mathcal{L}BG$. If we equipped BG with the trivial S^1-action then s is S^1-equivariant. The problem is that we don't know how to define $s_!$ directly. Instead, in the proof of Theorem 54, we define $(EG\times_G\eta)_!$. And using a simplicial model of EG, we give S^1-actions on BG and $EG\times_G G^{ad}$ such that $EG\times_G\eta:BG\hookrightarrow EG\times_G G^{ad}$ is S^1-equivariant.

10. A STRING BRACKET IN COHOMOLOGY

In this section,

-We show that the Δ operators of the Batalin-Vilkovisky algebras given by Corollaries 36 and 41, coincide with the Δ operator induced by the action of S^1 on $\mathcal{L}X$ (Proposition 60)

-from the Batalin-Vilkovisky algebra given by Corollaries 36 and 41, we define a Lie bracket on the S^1-equivariant cohomology $H^*_{S^1}(\mathcal{L}X)$ when X satisfying the hypothesis of the main theorem (Theorem 65). The definition of this string bracket in S^1-equivariant cohomology is similar but not identical to the definition of the Chas-Sullivan string bracket in homology (Theorem 64).

-from the Batalin-Vilkovisky algebra given by our inspirational theorem in section 2 when G is a finite group or given by Theorem 54 when G is a connected compact Lie group, we define a Lie bracket on the cyclic cohomology $HC^*(S_*(G))$ of the singular chains on G (Theorem 67).

We consider

$$act : S^1 \times \mathcal{L}X \to \mathcal{L}X$$

the reparametrization map defined by $act(\theta, \gamma(-)) := \gamma(- + \theta)$. Let $[S^1] \in H_1(S^1)$ be the fundamental class of the circle.

Proposition 60. *The operator $\Delta : H^*(\mathcal{L}X) \to H^{*-1}(\mathcal{L}X)$ of the Batalin-Vilkovisky algebras given by Corollaries 36 and 41, is the dual of the composite*

$$H_*(\mathcal{L}X) \overset{[S^1]\otimes -}{\to} H_*(S^1) \otimes H_*(\mathcal{L}X) \overset{act_*}{\to} H_*(\mathcal{L}X), \quad x \mapsto act_*([S^1] \otimes x).$$

Proof. For $\varepsilon = 0$ or 1, let $i_\varepsilon : S^1 \hookrightarrow S^1 \times [0,1]$, $x \mapsto (x, \varepsilon)$ be the two canonical inclusions. Consider the cylinder $C := S^1 \times [0,1]$ as the cobordism $F_{0,1+1}$:

$$S^1 \overset{i_0}{\hookrightarrow} S^1 \times [0,1] \overset{i_1}{\hookleftarrow} S^1.$$

Let $\mu(C) : H_*BDiff^+(C, \partial) \otimes H_*(\mathcal{L}X) \to H_*(\mathcal{L}X)$ be the evaluation map associated to C.

Recall from Section 6.4, that there is a canonical homotopy equivalence

$$f\mathcal{D}_2(1) \overset{\approx}{\to} BDiff^+(C, \partial).$$

It is also easy to construct a canonical homotopy equivalence $S^1 \overset{\approx}{\to} f\mathcal{D}_2(1)$. Denote by $B(\sigma) : S^1 \overset{\approx}{\to} BDiff^+(C, \partial)$ the composite of these two homotopy equivalences.

The operator $\Delta : H^*(\mathcal{L}X) \to H^{*-1}(\mathcal{L}X)$ of the Batalin-Vilkovisky algebras given by Corollaries 36 and 41, is by definition the dual of the composite

$$H_*(\mathcal{L}X) \overset{[S^1]\otimes -}{\to} H_*(S^1) \otimes H_*(\mathcal{L}X) \overset{B(\sigma)_* \otimes Id}{\to} H_*(BDiff^+(C, \partial)) \otimes H_*(\mathcal{L}X) \overset{\mu(C)}{\to} H_*(\mathcal{L}X).$$

By definition, ρ_{in} is $EDiff^+(C, \partial) \times_{Diff^+(C,\partial)} map(i_0, X)$. Since

$$map(i_0, X) : map(C, X) \overset{\approx}{\to} \mathcal{L}X$$

is a homotopy equivalence, ρ_{in} is also a homotopy equivalence. Therefore the shriek of ρ_{in}, $\rho_{in!}$, is equal to the inverse of ρ_{in*}, ρ_{in*}^{-1} and

$$\mu(C) := \rho_{out*} \circ \rho_{in!} = \rho_{out*} \circ \rho_{in*}^{-1}.$$

We identify S^1 with $\mathbb{R}/\mathbb{Z}$. The Dehn twist $D \in Diff^+(C, \partial)$ defined by $D(\theta, a) = (\theta + a, a)$, is the generator of the mapping class group $\Gamma_{0,1+1} = \pi_0(Diff^+(C, \partial)) \cong \mathbb{Z}$. By [22], the morphism of groups $\sigma : \mathbb{Z} \overset{\approx}{\to} Diff^+(C, \partial)$ sending $n \in \mathbb{Z}$ to the n-th composite D^n of D, is a homotopy equivalence. By applying the classifying construction, we obtain the map that we denoted before $B(\sigma)$. The morphism of groups σ induces the commutative diagram

$$\begin{tikzcd}
BDiff^+(C,\partial) \times \mathcal{L}X & EDiff^+(C,\partial) \times_{Diff^+(C,\partial)} map(C,X) \arrow[l, "\rho_{in}"', "\approx"] \arrow[r, "\rho_{out}"] & \mathcal{L}X \\
\mathbb{R}/\mathbb{Z} \times \mathcal{L}X \arrow[u, "B(\sigma)\times\mathcal{L}X", "\approx"'] & \mathbb{R} \times_{\mathbb{Z}} map(C,X) \arrow[l, "r_0", "\approx"'] \arrow[r, "r_1"] \arrow[u, "E(\sigma)\times_\sigma map(C,X)", "\approx"'] & \mathcal{L}X \arrow[u, "Id"']
\end{tikzcd}$$

where r_0 and r_1 are the maps defined by $r_0([\theta,f]) := ([\theta], f(-,0))$ and $r_1([\theta,f]) := f\circ i_1 = f(-,1)$ for $\theta\in\mathbb{R}$ and $f\in map(C,X)$. Since r_0 is equal to $\mathbb{R}\times_{\mathbb{Z}} map(i_0,X)$, r_0 is also a homotopy equivalence. Therefore by the commutativity of the above diagram

$$\mu(C)\circ(B(\sigma)_*\otimes H_*(\mathcal{L}X)) = \rho_{out*}\circ\rho_{in*}^{-1}\circ(B(\sigma)_*\otimes H_*(\mathcal{L}X)) = r_{1*}\circ r_{0*}^{-1}.$$

Let $\Phi : I\times\mathbb{R}\times map(C,X)\to\mathcal{L}X$ by the map defined by $\Phi(t,\theta,f(-,-)) := f(-+t\theta, 1-t)$. Since $\Phi(t,\theta+n,f(-,-)) = \Phi(t,\theta,f\circ D^{-n}(-,-))$ for $n\in\mathbb{Z}$, Φ induces a well-defined homotopy $\overline{\Phi} : I\times\mathbb{R}\times_{\mathbb{Z}} map(C,X)\to\mathcal{L}X$ between r_1 and $act\circ r_0$. Therefore $\mu(C)\circ(B(\sigma)_*\otimes H_*(\mathcal{L}X)) = r_{1*}\circ r_{0*}^{-1} = act_*$. □

Proposition 61. *Let G be a topological group. Let $p : E \twoheadrightarrow B$ be a G-principal bundle (or more generally a G-Serre fibration in the sense of* [24, p. 28]*). Then p is a Serre fibration. Suppose that B is path-connected and that p is oriented with orientation class $w\in H_n(G)\cong H_n(p^{-1}(p(*))$. Then the composite*

$$p_!\circ H_*(p) : H_*(E)\to H_*(B)\to H_{*+n}(E)$$

is given by the action of $w\in H_n(G)$ on $H_*(E)$.

Proof. Consider the pull-back

$$\begin{tikzcd} G\times E \arrow[r, "action"] \arrow[d, "proj_1"'] & E \arrow[d, "p"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

where $proj_1$ is the projection on the first factor and *action* is the action of G on E. By naturality with respect to pull-backs,

$$p_!\circ H_*(p) = H_*(action)\circ proj_{1!}.$$

Let $\varepsilon : G\to *$ be the constant map to a point. If we orient ε with the orientation class $w\in H_n(G)$, $\varepsilon_!(*) = w$. Since $proj_1 = \varepsilon\times id_G$, $proj_{1!} = \varepsilon_!(*)\otimes id_{G!} = w\otimes id$. So finally,

$$p_!\circ H_*(p)(a) = H_*(action)(w\otimes a).$$

Under the assumption that G is path-connected, an alternative proof is to interpret $H_*(p)$ as an edge homomorphism [73, XIII.(7.2)] and to use that the Serre spectral is a spectral sequence of $H_*(G)$-modules. □

By propositions 61 and 60, we have:

Corollary 62. *Consider the S^1-principal bundle $p : ES^1\times\mathcal{L}X\to ES^1\times_{S^1}\mathcal{L}X$. The composite $p_!\circ H_*(p)$ coincides with the operator Δ.*

Lemma 63. *Let $\varepsilon\in\mathbb{Z}$ be an integer. Let $\mathbb{H}$ be a Batalin-Vilkovisky algebra (not necessarily with an unit) and $\mathcal{H}$ be a graded module. Consider a long exact sequence of the form*

$$\cdots\to\mathbb{H}_n\overset{E}{\to}\mathcal{H}_{n+\varepsilon}\to\mathcal{H}_{n+\varepsilon-2}\overset{M}{\to}\mathbb{H}_{n-1}\to\cdots$$

Suppose that the operator $\Delta : \mathbb{H}_i\to\mathbb{H}_{i+1}$ is equal to $M\circ E$. Then

$$\{a,b\} := (-1)^{|a|}E\left(M(a)\cup M(b)\right),\quad\forall a,b\in\mathcal{H}$$

defines a Lie bracket of degree $2-\varepsilon$ on $\mathcal{H}$ such that the morphism of degree $1-\varepsilon$, $M : \mathcal{H}_n\to\mathbb{H}_{n+1-\varepsilon}$ is a morphism of graded Lie algebras:

$$\{M(a),M(b)\} = (-1)^{1-\varepsilon}M(\{a,b\}).$$

Proof. Case $\varepsilon = 0$. The Lie algebra structure is proved by Chas and Sullivan in the proof of [12, Theorem 6.1]. In their proof, the relation

$$\{M(a), M(b)\} = (-1)^{|a|+1} M \circ E\,(M(a) \cup M(b)) = -M(\{a,b\}).$$

appears (See also [72, p. 136]).

General Case $\varepsilon \neq 0$. If we replace $\mathcal{H}$ by its desuspension $s^{-\varepsilon}\mathcal{H}$, we can apply the case $\varepsilon = 0$ since $(s^{-\varepsilon}\mathcal{H})_n = \mathcal{H}_{n+\varepsilon}$. Alternatively, to check the signs carefully, it is simpler to generalise the computations of Chas and Sullivan. $\square$

Theorem 64. [12, Theorem 6.1] *Let M be a compact oriented smooth manifold of dimension d. Then*

$$\{a,b\} := (-1)^{|a|-d} H_*(p)\,(p_!(a) \cup p_!(b))\,, \quad \forall a,b \in H_*^{S^1}(\mathcal{L}M)$$

defines a Lie bracket of degree $2-d$ on $H_*^{S^1}(\mathcal{L}M)$ such that

$$p_! : H_*^{S^1}(\mathcal{L}M) \to H_{*+1}(\mathcal{L}M)$$

is a morphism of Lie algebras (between the string bracket and the loop bracket).

Proof. By [12, Theorem 5.4], $\mathbb{H}_n := H_{n+d}(\mathcal{L}M)$ is a Batalin-Vilkovisky algebra. Consider the Gysin exact sequence in homology

$$\cdots \to H_{n+d}(\mathcal{L}M) \overset{H_{n+d}(p)}{\to} H^{S^1}_{n+d}(\mathcal{L}M) \to H^{S^1}_{n+d-2}(\mathcal{L}M) \overset{p_!}{\to} H_{n+d-1}(\mathcal{L}M) \to \cdots$$

By Corollary 62, we can apply Lemma 63 to it in the case $\varepsilon = 0$. $\square$

Theorem 65. *Let X be a space satisfying the hypothesis of Corollary 36 or of Corollary 41. Denote by d the top degre of $H_*(\Omega X)$. Then*

$$\{a,b\} := (-1)^{|a|-d} p^!\,(H^*(p)(a) \cup H^*(p)(b))\,, \quad \forall a,b \in H^*_{S^1}(\mathcal{L}X)$$

defines a Lie bracket of (upper) degree $-1-d$ on $H^*_{S^1}(\mathcal{L}X)$ such that

$$H^*(p) : H^*_{S^1}(\mathcal{L}X) \to H^*(\mathcal{L}X)$$

is a morphism of Lie algebras.

Proof. By Corollaries 36 or 41, $\mathbb{H}_{-n} := H^{n+d}(\mathcal{L}X)$ is a Batalin-Vilkovisky algebra not necessarily with an unit. Consider the Gysin exact sequence in cohomology

$$\cdots \to H^{n+d}(\mathcal{L}X) \overset{p^!}{\to} H^{n+d-1}_{S^1}(\mathcal{L}X) \to H^{n+d+1}_{S^1}(\mathcal{L}X) \overset{H^{n+d+1}(p)}{\to} H^{n+d+1}(\mathcal{L}X) \to \cdots$$

By Corollary 62, we can apply Lemma 63 to it in the case $\varepsilon = 1$. $\square$

Remark 66. Under the hypothesis of our main theorem, one can also define a Lie bracket of degree $2+d$ on the equivariant homology $H_*^{S^1}(\mathcal{L}X)$, exactly as Chas-Sullivan string bracket. But this bracket will often be zero since the product on $H_*(\mathcal{L}X)$ is often trivial.

Theorem 67. *a) Let G be a finite group. Then*

$$\{a,b\} := (-1)^{|a|}\partial\,(I(a) \cup I(b))\,, \quad \forall a,b \in HC^*(\mathbb{F}[G])$$

defines a Lie bracket of (upper) degree -1 on the cyclic cohomology of the group ring of G such that the composite

$$HC^*(\mathbb{F}[G]) \overset{I}{\to} HH^*(\mathbb{F}[G];\mathbb{F}[G]^\vee) \overset{\cong}{\to} HH^*(\mathbb{F}[G];\mathbb{F}[G])$$

is a morphism of Lie algebras.

b) Let G be a compact connected Lie group of dimension d. Then

$$\{a,b\} := (-1)^{|a|-d}\partial\left(I(a)\cup I(b)\right), \quad \forall a,b\in HC^*(S_*(G))$$

defines a Lie bracket of (upper) degree $-1-d$ on the cyclic cohomology of the algebra of singular chains on G such that the composite

$$HC^{*+d}(S_*(G)) \xrightarrow{I} HH^{*+d}(S_*(G);S^*(G)) \underset{\cong}{\xrightarrow{\mathbb{D}}} HH^*(S_*(G);S_*(G))$$

is a morphism of Lie algebras.

Proof. For a), let $A := \mathbb{F}[G]$ and $d := 0$. For b), let $A := S_*(G)$. By our inspirational theorem in section 2 or Theorem 54, $\mathbb{H}_{-n} := HH^{n+d}(A;A^\vee)$ is a Batalin-Vilkovisky algebra (with an unit).

Consider Connes long exact sequence in homology [44, 2.2.1]

$$\cdots \to HH_{n+d}(A;A) \xrightarrow{I} HC_{n+d}(A) \xrightarrow{S} HC_{n+d-2}(A) \xrightarrow{\partial} H_{n+d-1}(A;A) \to \cdots$$

Usually, the map ∂ is unfortunately denoted B. The composite

$$HH_n(A;A) \xrightarrow{I} HC_n(A) \xrightarrow{\partial} HH_{n+1}(A;A)$$

coincides with Connes boundary map $H_*(B)$ (See [71, Notational consistency, p. 348-9] or [52, proof of Proposition 7.1] where the mixed complex considered should be the Hochschild chain complex of A). By dualizing, we have Connes long exact sequence in cohomology [44, 2.4.4]

$$\cdots \to HH^{n+d}(A;A^\vee) \xrightarrow{\partial} HC^{n+d-1}(A) \xrightarrow{S} HC^{n+d+1}(A) \xrightarrow{I} H^{n+d+1}(A;A^\vee) \to \cdots$$

Since $H(B^\vee) = I\circ\partial$, we can apply Lemma 63 to it in the case $\varepsilon = 1$. □

In part a) of Theorem 67, the group ring $\mathbb{F}[G]$ can be replaced by any symmetric Frobenius algebra A. In [52, Corollary 1.5], the second author defines a Lie bracket of (upper) degree -2 on the negative cyclic cohomology $HC^*_-(A)$ of any symmetric Frobenius algebra A.

Let M be a simply-connected manifold of dimension d. Let $S^*(M)$ be the algebra of singular cochains on M. In [53, Corollary 23], the second author defines a Lie bracket of lower degree $2-d$ on the negative cyclic cohomology $HC^*_-(S^*(M))$. In [53, Conjecture 24], the second author conjectures that the Jones isomorphism is an isomorphism of graded Lie algebras between this bracket and the Chas-Sullivan bracket on $H^{S^1}_*(LM)$. Dually, we conjecture

Conjecture 68. *Let G be a connected compact Lie group of dimension d.*

i)The composite of the isomorphism due to Burghelea, Fiedorowicz and Goodwillie [11, 32]

$$H^{*+d}(LBG) \xrightarrow{\cong} HH^{*+d}(S_*(G),S^*(G))$$

and of the isomorphism given by Theorem 54

$$\mathbb{D} : HH^{*+d}(S_*(G);S^*(G)) \xrightarrow{\cong} HH^*(S_*(G);S_*(G))$$

is a morphism of graded algebras between the algebra given by Corollary 36 and the underlying algebra on the Gerstenhaber algebra $HH^*(S_*(G);S_*(G))$.

ii)The isomorphism due to Burghelea, Fiedorowicz and Goodwillie [11, 32]

$$H^*_{S^1}(LBG) \xrightarrow{\cong} HC^*(S_*(G))$$

is a morphism of graded Lie algebras between the Lie brackets defined by Theorem 65 and Theorem 67.

Note that part i) of the conjecture implies part ii) (by the same arguments as in the last paragraph of [53]).

Following Freed, Hopkins and Teleman [27, 26] in twisted equivariant K-theory ${}^{\tau}K^*_G(G^{ad})$, one can easily define a *fusion product* on the equivariant cohomology $H^{*+d}_G(G^{ad}) \cong H^{*+d}(EG\times_G G^{ad}) \cong H^{*+d}(LBG)$ [35]. In [36], Gruher and Westerland gives an isomorphism of graded algebras between $H^{*+d}(LBG)$ and $HH^*(S_*(G);S_*(G))$ compatible with this fusion product. Note that to prove this isomorphism of algebras, they used the following theorem of Felix, the second author and Thomas.

Theorem 69. [25, Corollary 2] *Let X be a simply connected space such that $H_*(X)$ is finite dimensional in each degree. Then there is an isomorphism of Gerstenhaber algebras*

$$HH^*(S^*(X), S^*(X)) \cong HH^*(S_*(\Omega X), S_*(\Omega X)).$$

Here ΩX is the topological monoid of Moore pointed loops.

Let G be any topological group. By [24, Proposition 2.10 and Theorem 4.15], the differential graded algebras $S_*(G)$ and $S_*(\Omega BG)$ are weakly equivalent, therefore by [25, Theorem 3], there is an isomorphism of Gerstenhaber algebras

$$HH^*(S_*(G), S_*(G)) \cong HH^*(S_*(\Omega BG), S_*(\Omega BG)).$$

By applying Theorems 54 and 69, we obtain

Theorem 70. *Let G be a connected compact Lie group. Denote by $S^*(BG)$ the algebra of singular cochains on the classifying space of G. The Gerstenhaber algebra $HH^*(S^*(BG);S^*(BG))$ is a Batalin-Vilkovisky algebra.*

11. Appendix:(not) choosing an orientation class and signs problems

In this section, we explain that in fact, we do not choose an orientation class w_F for each cobordism F. Instead, we put all the possible choices of an orientation class in a prop, the prop $\mathrm{Det}H_1(F,\partial_{in}F;\mathbb{Z})$, to ensure the compatibility with gluing, disjoint union, etc .... This prop appeared first in [18] and [30]. As a consequence, we explain that in fact, in our main theorem for simply connected spaces, Theorem 33, $H_*(\mathcal{L}X)$ is a degree d (non-counital non-unital) homological conformal field theory.

11.1. **Integration along the fiber without orienting.**

Let $F \hookrightarrow E \overset{p}{\twoheadrightarrow} B$ an orientable fibration that we don't orient for the moment. The Serre spectral sequence (Compare with Section 2.2.1) gives the linear application of degree 0

$$\int_p : H_n(F;\mathbb{F}) \otimes H_*(B;\mathbb{F}) \to H_{*+n}(E;\mathbb{F})$$

which is independant of any choice of orientation class. If we choose an orientation class $w \in H_n(F;\mathbb{F})$, then we have an oriented fibration p whose integration along the fibre $p_!$ is given by $p_!(b) := \int_p (w \otimes b)$ for any $b \in H_*(B;\mathbb{F})$.

11.2. **The prop $H_{-d\chi(F)}(map_*(F_{p+q}/\partial_{in}F,X))$.** Let X be a simply connected space such that $H_*(\Omega X)$ is finite dimensional. In Section 2.3, we explain the formulas that orientation classes must satisfy, in order for the integration along the fibre to be natural, compatible with composition and product. With theses rules, the family of direct sum of graded vector spaces

$$\bigoplus_{F_{p+q}} H_{-d\chi(F)}(map_*(F_{p+q}/\partial_{in}F,X);\mathbb{F})$$

form a $\mathbb{F}$-linear graded prop. Here the direct sum is taken over a set of representatives F_{p+q} of the oriented cobordisms classes from $\coprod_{i=1}^p S^1$ to $\coprod_{i=1}^q S^1$, whose path components have at least one incoming-boundary component and one outgoing-boundary component (Compare with the topological prop defined in Proposition 11). For example, let us explain what is the composition of this prop using the notations of Proposition 31 and of the paragraph Composition in Section 2.3.

Let $F_{g,p+q}$ and $F'_{g',q+r}$ be two oriented cobordisms. Let $F''_{g",p+r}$ be the oriented cobordism obtained by gluing. Consider the two orientable fibrations

$$f: map(F''_{g",p+r},X)\twoheadrightarrow map(F_{g,p+q},X)$$

given in the pull back (32), and

$$g:=map(in,X):map(F_{g,p+q},X)\twoheadrightarrow \mathcal{L}X^{\times p}.$$

By pull back, we obtain an orientable fibration

$$f': map_*(F''_{g,p+r}/\partial_{in},X)\twoheadrightarrow map_*(F_{g,p+q}/\partial_{in},X)$$

with fibre $map_*(F'_{g',q+r}/\partial_{in},X)$. Therefore the Serre spectral sequence gives the linear isomorphism of degree 0

$$\int_{f'}: H_{-d\chi(F')}map_*(F'_{g',q+r}/\partial_{in},X)\otimes H_{-d\chi(F)}map_*(F_{g,p+q}/\partial_{in},X)\overset{\cong}{\to} H_{-d\chi(F'')}map_*(F''_{g,p+r}/\partial_{in},X)$$

which is the composition of the prop. Suppose that we choose an orientation class for f, $w_f\in H_{-d\chi(F')}map_*(F'_{g',q+r}/\partial_{in},X)$, and an orientation class for g, $w_g\in H_{-d\chi(F)}map_*(F_{g,p+q}/\partial_{in},X)$. Then $(g\circ f)_!$ is equal to $f_!\circ g_!$ if the orientation class chosen $w_{g\circ f}$ for $g\circ f$ is equal to $f'_!(w_g)$, i. e. to $\int_{f'}(w_f\otimes w_g)$(See paragraph Composition in Section 2.3).

Using $\int_{\rho_{in}}$ instead of $\rho_{in!}$, we obtain an linear evaluation product of degree 0

$$\int_F: H_{-d\chi(F)}map_*(F_{g,p+q}/\partial_{in},X)\otimes H_*(BDiff^+(F,\partial))\otimes H_*(\mathcal{L}X)^{\otimes p}\to H_*(\mathcal{L}X)^{\otimes q}.$$

If we choose an orientation class $w_F\in H_{-d\chi(F)}map_*(F_{g,p+q}/\partial_{in},X)$, of course this new evaluation product is related to the old one by $\mu(F)(a\otimes v)=\int_F(w_F\otimes a\otimes v)$ for any $a\otimes v\in H_*(BDiff^+(F,\partial))\otimes H_*(\mathcal{L}X)^{\otimes p}$.

The section 5 shows in fact that, with $\int_F$, $H_*(\mathcal{L}X)$ is an algebra over the tensor product of props

$$H_{-d\chi(F)}map_*(F_{g,p+q}/\partial_{in},X)\otimes H_*(BDiff^+(F,\partial)).$$

11.3. **The orientation of a finitely generated free $\mathbb{Z}$-module V, Det$V$.** Let V be a free abelian group of rank n. Let $\Theta = (e_1, \cdots, e_n)$ et $\Theta' = (e'_1, \cdots, e'_n)$ be two basis of V. Let $\varphi : V \to V$ be the $\mathbb{Z}$-linear automorphism sending e_i to e'_i. By definition, Θ and Θ' belong to the same orientation class if the determinant of φ, $det\varphi$, is equal to $+1$. Equivalently, the application induced by φ, on the n-th exterior powers, $\Lambda^n\varphi : \Lambda^n V \to \Lambda^n V$ is the identity map, this means that $e'_1 \wedge \cdots \wedge e'_n = e_1 \wedge \cdots \wedge e_n$. Therefore the application which maps the orientation class $[\Theta]$ of a basis $\Theta = (e_1, \cdots, e_n)$, on the generator $e_1 \wedge \cdots \wedge e_n$ of $\Lambda^n V \cong \mathbb{Z}$ is a bijection. So a choice of an generator of $\Lambda^n V \cong \mathbb{Z}$ is a choice of an orientation on V. Let us set

$$\det V := \Lambda^n V.$$

11.4. **The prop det$H_1(F, \partial_{in}; \mathbb{Z})$.** In [18, p. 183] and [30, Lemma 13], Costello and Godin explain that the familly of graded abelians groups

$$\bigoplus_{F_{p+q}} \det H_1(F, \partial_{in}; \mathbb{Z})$$

form a $\mathbb{Z}$-linear graded prop. Again, here the direct sum is taken over a set of representatives F_{p+q} of the oriented cobordisms classes from $\coprod_{i=1}^{p} S^1$ to $\coprod_{i=1}^{q} S^1$, whose path components have at least one incoming-boundary component and one outgoing-boundary component.

Let us explain what is the composition of this prop. Let $F_{g,p+q}$ and $F'_{g',q+r}$ be two oriented cobordisms. Let $F''_{g'',p+r}$ be the oriented cobordism obtained by gluing. By excision, $H_*(F''_{g'',p+r}, F_{g,p+q}) \cong H_*(F'_{g',q+r}, \partial_{in} F'_{g',q+r})$. By Proposition 14 $\tilde{H}_2(F'_{g',q+r}/\partial_{in} F'_{g',q+r}) = 0$ and $\tilde{H}_0(F_{g,p+q}/\partial_{in} F_{g,p+q}) = 0$. Therefore the long exact sequence associated to the triple $(F''_{g'',p+r}, F_{g,p+q}, \partial_{in} F_{g,p+q})$ reduces to the short exact sequence:

$$0 \longrightarrow H_1(F_{g,p+q}, \partial_{in} F_{g,p+q}) \longrightarrow H_1(F''_{g'',p+r}, \partial_{in} F_{g,p+q}) \longrightarrow H_1(F'_{g',q+r}, \partial_{in} F'_{g',q+r}) \longrightarrow 0.$$

Godin's situation [30, (33)] is more complicated because she considers open-closed cobordisms and in this paper, we consider only closed cobordisms.

A short exact sequence of finite type free abelian groups $0 \longrightarrow U \longrightarrow V \longrightarrow W \longrightarrow 0.$ gives [17, Lemma 1 p. 1176]) a natural isomorphism $\det U \otimes \det V \overset{\cong}{\to} \det W$. Therefore, we have a canonical isomorphism

$$\det H_1(F_{g,p+q}, \partial_{in} F_{g,p+q}; \mathbb{Z}) \otimes \det H_1(F'_{g',q+r}, \partial_{in} F'_{g',q+r}; \mathbb{Z}) \overset{\cong}{\to} \det H_1(F''_{g'',p+r}, \partial_{in} F''_{g'',p+q}; \mathbb{Z}).$$

This is the composition of the prop.

11.5. **The prop isomorphim det$H_1(F, \partial_{in}; \mathbb{Z})^{\otimes d} \otimes_{\mathbb{Z}} \mathbb{F} \cong H_{-d\chi(F)}(map_*(F/\partial_{in}, X); \mathbb{F})$.** Obviously, our conformal field theory structure on $H_*(\mathcal{L}X; \mathbb{F})$ when X is simply-connected topological space, depends of a choice of a generator $w \in H_d(\Omega X; \mathbb{F})$. So let us choose a fixed generator $w \in H_d(\Omega X; \mathbb{F})$.

Let F_{p+q} be an oriented cobordism whose path connected components have at least one incoming boundary component and also at least one outgoing component. By Proposition 14, the quotient space F/∂_{in} is homotopy equivalent to a wedge $\vee_{-\chi(F)} S^1$. Let $f : F/\partial_{in} \overset{\approx}{\to} \vee_{-\chi(F)} S^1$ be a pointed homotopy equivalence. Consider the composite of the Kunneth map, $Kunneth$, and of $H_*(map_*(f, X))$.

$$H_*(\Omega X)^{\otimes -\chi(F)} \underset{\cong}{\overset{Kunneth}{\to}} H_*(map_*(\vee_{-\chi(F)} S^1, X)) \underset{\cong}{\overset{H_*(map_*(f,X))}{\to}} H_*(map_*(F/\partial_{in}, X)).$$

Let w_f denote the image of $w^{\otimes -\chi(F)}$ by this isomorphism. On the other hand, let Θ_f be the image of the canonical basis of $\mathbb{Z}^{\times -\chi(F)}$ by the inverse of $H_1(f;\mathbb{Z})$:

$$\mathbb{Z}^{\times -\chi(F)} = H_1(\vee_{-\chi(F)}S^1;\mathbb{Z}) \overset{H_1(f;\mathbb{Z})^{-1}}{\to} H_1(F/\partial_{in};\mathbb{Z}) \cong H_1(F,\partial_{in};\mathbb{Z}).$$

Proposition 71. *Let f and $g : F/\partial_{in} \overset{\approx}{\to} \vee_{-\chi(F)}S^1$ be two pointed homotopy equivalences. Let w_f and w_g be the two corresponding generators of $H_{-d\chi(F)}(map_*(F/\partial_{in}, X);\mathbb{F})$. Let Θ_f and Θ_g be the two associated basis of $H_1(F,\partial_{in};\mathbb{Z})$. Then*

$$w_f = det_{\Theta_f}(\Theta_g)^d w_g,$$

where $det_{\Theta_f}(\Theta_g)$ is the d-th power of the determinant of the basis Θ_g with respect to the basis Θ_f.

Proof. Let $h : \vee_{-\chi(F)}S^1 \overset{\approx}{\to} \vee_{-\chi(F)}S^1$ be a pointed homotopy equivalence such that f is homotopic to the composite $h \circ g$. Since $H_1(f;\mathbb{Z}) = H_1(h;\mathbb{Z}) \circ H_1(g;\mathbb{Z})$,

$$\begin{aligned} det_{\Theta_f}(\Theta_g) &= det_{H_1(f;\mathbb{Z})(\Theta_f)} H_1(f;\mathbb{Z})(\Theta_g) \\ &= det_{(\text{canonical basis})} H_1(h;\mathbb{Z})(\text{canonical basis}) = det H_1(h;\mathbb{Z}) \end{aligned}$$

where canonical basis denotes the canonical basis of $\mathbb{Z}^{\times -\chi(F)}$. By Proposition 20,

$$\begin{aligned} w_f &:= H_*(map_*(f,X)) \circ Kunneth(w^{\otimes -\chi(F)}) \\ &= H_*(map_*(g,X)) \circ H_*(map_*(h,X)) \circ Kunneth(w^{\otimes -\chi(F)}) \\ &= det H_1(h;\mathbb{Z})^d H_*(map_*(g,X)) \circ Kunneth(w^{\otimes -\chi(F)}) = det H_1(h;\mathbb{Z})^d w_g \end{aligned}$$

□

Let $\Theta_f = (e_1, \cdots, e_{-\chi(F)})$ be the basis of $H_1(F,\partial_{in};\mathbb{Z})$ associated to a pointed homotopy equivalence $f : F/\partial_{in} \overset{\approx}{\to} \vee_{-\chi(F)}S^1$. As recalled in Section 11.3, the orientation class of Θ_f, $[\Theta_f]$, corresponds to the generator $e_1 \wedge \cdots \wedge e_{-\chi(F)}$ of $\det H_1(F,\partial_{in};\mathbb{Z})$. Therefore $[\Theta_f]^{\otimes d}$ is a generator of the tensor product $[\det H_1(F,\partial_{in};\mathbb{Z})]^{\otimes d}$. Consider the unique $\mathbb{Z}$-linear map

$$Or(F) : [\det H_1(F,\partial_{in};\mathbb{Z})]^{\otimes d} \to H_{-d\chi(F)}(map_*(F/\partial_{in}, X);\mathbb{F})$$

sending the generator $[\Theta_f]^{\otimes d}$ to the generator w_f.

Corollary 72. *The morphism $Or(F)$ is independant of the pointed homotopy equivalence $f : F/\partial_{in} \overset{\approx}{\to} \vee_{-\chi(F)}S^1$.*

Proof. Let $g : F/\partial_{in} \overset{\approx}{\to} \vee_{-\chi(F)}S^1$ be another pointed homotopy equivalence. Let Θ_g be the basis of $H_1(F,\partial_{in};\mathbb{Z})$ associated to g. By definition of orientation classes, $[\Theta_g] = det_{\Theta_f}(\Theta_g)[\Theta_f]$. Therefore, by Proposition 71,

$$Or(F)([\Theta_g]^{\otimes d}) = Or(F)\left(det_{\Theta_f}(\Theta_g)^d[\Theta_f]^{\otimes d}\right) = det_{\Theta_f}(\Theta_g)^d w_f = w_g.$$

□

We claim [2] that the family of $\mathbb{Z}$-linear map

$$Or(F) : \det H_1(F,\partial_{in};\mathbb{Z})]^{\otimes d} \to H_{-d\chi(F)}(map_*(F/\partial_{in}, X);\mathbb{F})$$

[2]Guldberg checks this claim in Section 2.2.2. of his 2011 masterthesis.

gives a morphism of props from the tensor product of prop $\det H_1(F,\partial_{in};\mathbb{Z})^{\otimes d}$ to the prop $H_{-d\chi(F)}(map_*(F/\partial_{in},X);\mathbb{F})$. At the end of section 11.2, we explain that $H_*(\mathcal{L}X)$ is an algebra over the tensor product of props

$$H_{-d\chi(F)}map_*(F_{g,p+q}/\partial_{in},X)\otimes_{\mathbb{F}} H_*(BDiff^+(F,\partial)).$$

Therefore $H_*(\mathcal{L}X)$ is an algebra over the tensor product of props

$$\det H_1(F,\partial_{in};\mathbb{Z})^{\otimes d}\otimes_{\mathbb{Z}} H_*(BDiff^+(F,\partial)).$$

That is $H_*(\mathcal{L}X)$ is a *d-dimensional (non-unital non-counital) homological conformal field theory* (in the sense of [18, Definition p. 169] or [30, Definition 3, Section 4.1]).

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