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Gerstenhaber duality in Hochschild cohomology

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Abstract

Let C be a differential graded chain coalgebra, $\overline{\Omega}C$ the reduced cobar construction on C and $C^\vee$ the dual algebra. We prove that for a large class of coalgebras C there is a natural isomorphism of Gerstenhaber algebras between the Hochschild cohomologies $HH^*(C^\vee, C^\vee)$ and $HH^*(\overline{\Omega}C; \overline{\Omega}C)$. This result yields to a Hodge decomposition of the loop space homology of a closed oriented manifold, when the field of coefficients is of characteristic zero.

MSC: 16E40; 17A30; 17B55; 81T30; 55P35

1. Introduction

Let $\Bbbk$ be a field, C be a chain coalgebra and $A = C^\vee$ be its dual cochain algebra. The purpose of this text is to compare the Gerstenhaber algebra structures on the Hochschild cohomologies $HH^*(A; A)$ and $HH^*(\bar{\Omega}C; \bar{\Omega}C)$, where $\bar{\Omega}$ denotes the reduced cobar construction.

We will always suppose that C is supplemented, $C = \Bbbk \oplus \overline{C}$, that $H(C)$ is finitely generated, and that $H(\overline{C}) = H(\overline{C})_{\geqslant 2}$.

Our first result reads

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Theorem 1. *With the above notation, there exists an isomorphism of Gerstenhaber algebras*

$$\mathscr{D}_{\mathscr{C}} : HH^*(\overline{\Omega}C; \overline{\Omega}C) \longrightarrow HH^*(A; A).$$

The definition of the Hochschild cohomology is given in Section 2, and the description of the morphism $\mathscr{D}_{\mathscr{C}}$ in Section 5. Our proof of Theorem 1 relies heavily on properties of quasi-free algebras and on the structure of Lie algebras of derivations.

A *quasi-free algebra* is a tensor algebra *TV* together with a differential *d* such that *V* is the union $V = \cup V(k)$ of an increasing family of subspaces $V(0) \subset V(1) \subset \cdots$ such that $d(V(0)) = 0$ and $d(V(k)) \subset T(V(k-1))$. For instance the cobar construction $\bar{\Omega}C$ is a quasi-free algebra. Observe also that for any differential graded algebra *A* there is a quasi-isomorphism of differential graded algebras $(T(V), d) \overset{\simeq}{\twoheadrightarrow} A$ with $(T(V), d)$ a quasi-free algebra [6, Proposition 3.1]. The algebra $(T(V), d)$ is then called a quasi-free model of *A*.

Two differential graded Lie algebras are naturally associated to the cochain algebra *A*, the differential graded Lie algebra of derivations of *A*, denoted Der A, with the commutator bracket $[-, -]$ and the differential $D = [d, -]$, and the differential graded Lie algebra $\widetilde{\mathrm{Der}}\, A = \mathrm{Der}\, A \oplus sA$

$$D(\theta + sx) = D(\theta) - ad_x - sd(x), \quad ad_x(y) = xy - (-1)^{|x||y|}yx,$$
$$[\theta, \theta' + sx] = [\theta, \theta'] + (-1)^{|\theta|}s\theta(x), \quad [sx, sy] = 0,$$

with $\theta, \theta' \in \mathrm{Der}\, A$, $x, y \in A$ and $(sA)_i = A_{i-1}$.

The next result is very useful for computations and is a corner stone in the proof of Theorem 1.

Theorem 2. *Let $A=(TV, d)$ be a quasi-free algebra. Then there exists quasi-isomorphisms of differential graded Lie algebras*

$$s\mathbf{C}^*(A; A) \overset{\simeq}{\longleftarrow} \widetilde{\mathrm{Der}}\, A \overset{\simeq}{\longrightarrow} \mathrm{Der}\, \widetilde{A},$$

where $\mathbf{C}^*(A; A)$ denotes the Hochschild cochain algebra of A and where $\widetilde{A}=(T(V \oplus \Bbbk\varepsilon), \tilde{d})$ with $|\varepsilon| = 1$, $\tilde{d}\varepsilon = \varepsilon^2$ and $\tilde{d}v = dv + \varepsilon v - (-1)^{|v|}v\varepsilon$, $v \in V$.

We directly deduce

Corollary 1. *Let $(TV, d) \overset{\simeq}{\to} A$ and $(TW, d) = \overline{\Omega}C$ be quasi-free models, then we have isomorphisms of graded Lie algebras*

$$H(\widetilde{\mathrm{Der}}(TW, d)) \cong sHH^*(A; A) \cong H(\widetilde{\mathrm{Der}}(TV, d)).$$

We remark here that an analogous result concerning derivations of a differential graded Lie algebra and derivations of its cochain algebra was pointed out by Schlessinger and Stasheff twenty years ago. [13].

An important property of the Hochschild cohomology is its behavior with respect to quasi-isomorphisms:

Theorem 3. *HH^* extends to a functor from the category whose objects are differential graded algebras and whose morphisms are quasi-isomorphisms to the category of Gerstenhaber algebras. Each quasi-isomorphism of differential graded algebras $f : A \to B$ induces an isomorphism $HH^*(f) : HH^*(A; A) \to HH^*(B; B)$, and $\mathscr{D}_C$ makes commutative the diagram*

$$\begin{array}{ccc} HH_*(\overline{\Omega}C;\overline{\Omega}C) & \overset{HH^*(\overline{\Omega}g)}{\longrightarrow} & HH_*(\overline{\Omega}C';\overline{\Omega}C') \\ \mathscr{D}_C \downarrow & & \downarrow \mathscr{D}_{C'} \\ HH_*(C^\vee;C^\vee) & \overset{HH^*(g^\vee)}{\longleftarrow} & HH^*(C'^\vee;C'^\vee). \end{array}$$

The fact that $HH^*(f)$ preserves the multiplication was already proved by Keller [10]. Theorem 3 shows that $HH^*(f)$ preserves also the Lie algebra structure, and this answers in particular a question of Keller [10].

Theorems 1 and 2 have a geometrical flavor coming from the recent work of Chas and Sullivan on the loop homology $\mathbb{H}_*(M^{S^1})$ of a closed orientable manifold M of dimension d [4]. For recall, the loop homology of M is the ordinary homology of the free loop space $LM = M^{S^1}$ with degrees shifted by d, i.e. $\mathbb{H}_*(LM) = H_{*+d}(LM)$. In [4], Chas and Sullivan define a product, called the loop product, and a Lie bracket, called the loop bracket, on $\mathbb{H}_*(LM)$ making $\mathbb{H}(LM)$ into a Gerstenhaber algebra (see also [7]).

The relation with Hochschild cohomology is due to Cohen and Jones who construct an isomorphism of algebras $J : \mathbb{H}_*(LM) \to HH^*(\mathscr{C}^*(M); \mathscr{C}^*(M))$, [5, Corollary 10]. It is then natural to conjecture that J is an isomorphism of Gerstenhaber algebras.

Corollary 2. *Let M be a simply-connected closed oriented manifold. If J is an isomorphism of Gerstenhaber algebras, then the Gerstenhaber algebra $\mathbb{H}_*(LM)$ is isomorphic to the Gerstenhaber algebra $HH^*(\mathscr{C}_*(\Omega M), \mathscr{C}_*(\Omega M))$.*

Here ΩM denotes the space of based loops on M, and $\mathscr{C}_*(M)$ (resp. $\mathscr{C}^*(M)$) denotes the normalized singular chain coalgebra (resp. cochain algebra) on M with coefficients in $\Bbbk$.

If $\Bbbk$ is a field of characteristic zero we can go further. The Adams–Hilton model of M is the universal enveloping algebra of a differential graded Lie algebra $(L, d) = (\mathbb{L}(V), d)$ [2]. Then

$$\begin{aligned} \mathbb{H}_*(LM) &\cong HH^*(\mathscr{C}_*(\Omega M); \mathscr{C}_*(\Omega M)) && \text{(Corollary 2)} \\ &\cong HH^*(U(L,d); U(L,d)) && \text{(by naturality)} \\ &\cong \mathrm{Ext}_{U(L,d)^e}(U(L,d), U(L,d)) && \\ &\cong \mathrm{Ext}_{U(L,d)}(\Bbbk, U(L,d)) && \text{[3, Theorem XIII.5.1],} \end{aligned}$$

where UL is considered as an UL-module via the adjoint representation. Denote by $\Gamma^n(V)$ the vector space generated by the elements $\sum_{\sigma\in\Sigma_n} v_{\sigma(1)}\cdots v_{\sigma(n)}$, with $v_i \in V$. Then the vector spaces $\Gamma^n(V)$ are stable under the adjoint action of UL and $UL \cong \oplus_{n\geqslant 0}\Gamma^n(V)$. This gives the "Hodge decomposition":

Corollary 3. *Under the above hypothesis there exists isomorphisms of graded vector spaces*

$$\mathbb{H}_*(LM) = \mathrm{Ext}_{U(L,d)}(\Bbbk, U(L,d)) = \oplus_{n\geqslant 0}\mathrm{Ext}_{UL}(\Bbbk, \Gamma^n(V)).$$

Theorem 1 and 2 give an explicit procedure to compute the Hochschild cohomology $HH^*(\mathscr{C}^*M; \mathscr{C}^*M)$. We illustrate this point by giving the Lie algebra structure of $HH^*(\mathscr{C}^*M; \mathscr{C}^*M)$ when $M = S^2$ and $M = \mathbb{C}P^2$.

The paper is organized as follows:

(2) Bar construction and Hochschild cohomology.
(3) Naturality of the Hochschild cohomology with respect to quasi-isomorphisms.
(4) Hochschild cohomology and derivations.
(5) The duality map $\mathscr{D}_C$ when C is a finite type coalgebra.
(6) The duality map $\mathscr{D}_C$ when $H(C)$ is finite type.
(7) Examples.

2. Bar construction and Hochschild cohomology

2.1. Notation

In this paper $\Bbbk$ will be a field. If $V=\{V_i\}_{i\in\mathbb{Z}}$ is a lower graded $\Bbbk$-module then $(sV)_n=V_{n-1}$, TV denotes the tensor algebra on V, and $TC(V)$ the free supplemented coalgebra generated by V. When we need upper graded $\Bbbk$-module we put $V_i = V^{-i}$ as usual.

Since we work with graded differential objects, we have to take care to signs. Recall that if $M = \{M_i\}_{i\in \mathrm{Z}}$ and $N = \{N_i\}_{i\in \mathrm{Z}}$ are differential graded $\Bbbk$-modules then

(a) $M \otimes N$ is a differential graded $\Bbbk$-module: $(M\otimes N)_r = \oplus_{p+q=r} M_p \otimes N_q$, $d_{M\otimes N} = d_M \otimes id_N + id_M \otimes d_N$,
(b) $\mathrm{Hom}(M, N)$ is a differential graded $\Bbbk$-module: $\mathrm{Hom}_n(M,N) = \prod_{k-l=n}\mathrm{Hom}(M_l, N_k)$, $Df = d_M \circ f - (-1)^{|f|} f \circ d_N$,
(c) the commutator, $[f,g] = f\circ g - (-1)^{|f||g|} g\circ f$, gives to the differential graded $\Bbbk$-module $\mathrm{End}(M) = \mathrm{Hom}(M, M)$ a structure of differential graded Lie algebra,
(d) if C is a differential graded coalgebra with diagonal Δ and A is a differential graded algebra with product μ then the cup product, $f \cup g = \mu \circ (f \otimes g) \circ \Delta$, gives to the differential graded $\Bbbk$-module $\mathrm{Hom}(C, A)$ a structure of differential graded algebra.

2.2. Bar construction

Let (A, d) be a differential graded supplemented algebra, $A = \Bbbk \oplus \overline{A}$, let (M, d) be a right differential graded A-module and let (N, d) be a left differential graded A-module.

The *two-sided bar constructions,* $\mathbb{B}(M; A; N)$ and $\overline{\mathbb{B}}(M; A; N)$ are defined as follows:

$$\mathbb{B}_k(M; A; N) = M \otimes T^k(sA) \otimes N, \quad \overline{\mathbb{B}}_k(M; A; N) = M \otimes T^k(s\overline{A}) \otimes N$$

A generic element is written $m[a_1|a_2|\ldots|a_k]n$ with degree $|m| + |n| + \sum_{i=1}^{k}(|sa_i|)$. The differential d is defined by

$$\begin{aligned} d(m[a_1|a_2|\ldots|a_k]n) = d(m)[a_1|a_2|\ldots|a_k]n \\ & - \sum_{i=1}^{k}(-1)^{\varepsilon_i} m[a_1|a_2|\ldots|d(a_i)|\ldots|a_k]n \\ & + (-1)^{\varepsilon_{k+1}} m[a_1|a_2|\ldots|a_k]d(n) \\ & + \sum_{i=2}^{k}(-1)^{\varepsilon_i} m[a_1|a_2|\ldots|a_{i-1}a_i|\ldots|a_k]n \\ & + (-1)^{|m|} ma_1[a_2|\ldots|a_k]n - (-1)^{\varepsilon_k} \\ & \times m[a_1|a_2|\ldots|a_{k-1}]a_k n \end{aligned}$$

with $\varepsilon_i = |m| + \sum_{j<i}(|sa_j|)$.

Hereafter we will consider the *normalized* and the *non-unital* bar constructions on A:

$$\overline{\mathbb{B}}(A) = \overline{\mathbb{B}}(\Bbbk; A; \Bbbk) = (TC(s\overline{A}), \overline{d}), \quad \widetilde{\mathbb{B}}(A) = \overline{\mathbb{B}}(\Bbbk \oplus A) = (TC(sA), \tilde{d}).$$

(In the latter formula A is considered as a non-unital algebra [12, p. 142]). The differentials $\overline{d}$ and $\overline{d}$ are given by the same formula

$$\begin{aligned} d([a_1|a_2|\cdots|a_k]) = & - \sum_{i=1}^{k}(-1)^{\varepsilon_i}[a_1|a_2|\ldots|d(a_i)|\ldots|a_k] \\ & + \sum_{i=2}^{k}(-1)^{\varepsilon_i}[a_1|a_2|\ldots|a_{i-1}a_i|\ldots|a_k]. \end{aligned}$$

An interesting property of the bar construction is its semifreeness. Recall that a differential R-module N is called semifree if N is the union of an increasing sequence of sub-modules $N(i)$, $i \geqslant 0$, such that each $N(i)/N(i-1)$ is an R-free module on a basis of cycles [6].

Proposition 2.3 (*6, Lemma 4.3*). *The canonical map* $\varphi: \mathbb{B}(A, A, A) \to A$ *defined by* $\varphi[] = 1$ *and* $\varphi([a_1|\cdots|a_k]) = 0$ *if* $k > 0$, *is a semifree resolution of* A *as an* A^e*-module*.

2.4 Hochschild cohomology. Let A be a supplemented differential graded algebra, $A^e := A \otimes A^{op}$ be its enveloping algebra, and M be a differential graded A-bimodule, i.e. an A^e-module. The *Hochschild cochain complex* of A with coefficients in M is the differential module

$$\mathbf{C}^*(A; M) = \mathrm{Hom}_{A^e}(\mathbb{B}(A; A; A), M).$$

Remark that the canonical isomorphism of graded modules $\Phi_{A,M} : \mathrm{Hom}_{A^e}(\mathbb{B}(A; A; A), M) \to \mathrm{Hom}(T(sA), M)$ defined by

$$\Phi_{A,M}(f) = (1_A[a_1|\dots|a_k]1_A \mapsto f([a_1|\dots|a_k]))$$

carries a differential $D_0 + D_1$ on $\mathrm{Hom}(T(sA), M)$ making $\Phi_{A,M}$ into an isomorphism of complexes. More explicitly, if $f \in \mathrm{Hom}(T(sA), M)$, we have

$$\begin{aligned} D_0(f)([a_1|a_2|\dots|a_k]) = d_M(f([a_1|a_2|\dots|a_k])) & \\ + \sum_{i=1}^{k} (-1)^{\bar{\varepsilon}_i} f([a_1|\dots|da_i|\dots|a_k]) & \\ - \sum_{i=2}^{k} (-1)^{\bar{\varepsilon}_i} f([a_1|\dots|a_{i-1}a_i|\dots|a_k]) & \end{aligned}$$

and

$$\begin{aligned} D_1(f)([a_1|a_2|\dots|a_k]) &= -(-1)^{|sa_1||f|} a_1 f([a_2|\dots|a_k]) \\ &\quad + (-1)^{\bar{\varepsilon}_k} f([a_1|a_2|\dots|a_{k-1}])a_k, \end{aligned}$$

where $\bar{\varepsilon}_i = |f| + |sa_1| + |sa_2| + \dots + |sa_{i-1}|$. We remark that D_0 is the usual differential on $\mathrm{Hom}(T(sA), M) = \mathrm{Hom}(\widetilde{\mathbb{B}}(A), M)$.

The *Hochschild cohomology of A with coefficients in M* is

$$HH^*(A; M) = H(\mathbf{C}^*(A; M)) \cong H((\mathrm{Hom}(TC(sA), M), D_0 + D_1)).$$

Let $\varphi : A \to A'$ be a homomorphism of differential graded algebras. Then A' is a differential graded A-bimodule via φ. Then the coalgebra structure on $T(sA)$ and the multiplication on A' makes $\mathbf{C}^*(A; A') \cong \mathrm{Hom}(T(sA), A')$ a differential graded algebra.

Denote by $ex_A : \mathrm{Hom}(T(sA), sA) \longrightarrow \mathrm{Coder}(T(sA))$ the linear isomorphism extending each linear map into a coderivation. We consider the degree 1 isomorphism

$$\beta_A : \mathbf{C}^*(A; A) \cong \mathrm{Hom}(TC(sA), A) \xrightarrow{\mathrm{Hom}(T(sA),s)} \mathrm{Hom}(T(sA), sA) \xrightarrow{ex_A} \mathrm{Coder}(\widetilde{\mathbb{B}}A).$$

Proposition 2.5 (*Gestenhaber [8], Stasheff [14]*). *The isomorphism β_A satisfies $(D_0 + D_1)\beta_A = -D\beta_A$, and is therefore an isomorphism of complexes.*

Observe that for $g \in \mathrm{Hom}(TC(sA), A)$

$$\beta_A(g)[a_1|\dots|a_n] = \sum_{0 \leqslant i \leqslant j \leqslant n} (-1)^{\mu}[a_1|\dots|a_i|g([a_{i+1}|\dots|a_j]) \\ |a_{j+1}|\dots a_n], \text{ with } \mu = |s \circ g|(|sa_1| + \dots + |sa_i|).$$

2.6 Gerstenhaber algebra. A *(graded) Gerstenhaber algebra* is a commutative graded algebra $G = \{G\}_{i \in \mathbb{Z}}$ with a degree 1 linear map

$$G_i \otimes G_j \to G_{i+j+1}, \quad x \otimes y \mapsto \{x, y\}$$

such that

(a) the suspension of G is a graded Lie algebra with bracket

$$(sG)_i \otimes (sG)_j \to (sG)_{i+j}, \quad sx \otimes sy \mapsto [sx, sy] := s\{x, y\}$$

(b) the product is compatible with the bracket, i.e. for $a, b, c \in G$,

$$\{a, bc\} = \{a, b\}c + (-1)^{|b|(|a|+1)} b\{a, c\}.$$

Let A be a supplemented differential graded algebra. The Lie algebra structure on $\mathrm{Coder}(\widetilde{\mathbb{B}}A)$ defines via β_A a Lie algebra structure on $s\mathbf{C}^*(A, A) \cong \mathrm{Hom}(T(sA), sA)$. This structure combines with the multiplication to make $HH^*(A; A)$ a Gerstenhaber algebra [14].

Note that the Gerstenhaber bracket $[-, -]$ on $s\mathbf{C}^*(A; A)$ is given by the formula

$$[sf, sg] = (sf)\bar{\circ}(sg) - (-1)^{|sf||sg|}(sg)\bar{\circ}(sf) \quad \text{where } (sf)\bar{\circ}(sg) = s(f \circ \beta_A(g)).$$

2.7 Cobar construction. Let (C, d) be a supplemented differential graded coalgebra, (R, d) be a right C-comodule and (L, d) be a left C-module. The *two-sided cobar constructions*, $\Omega(R; C; L)$ and $\overline{\Omega}(R; C; L)$ are defined as follows:

$$\Omega(R; C; L) = (R \otimes T(s^{-1}C) \otimes L, d_0 + d_1),$$
$$\overline{\Omega}(R; C; L) = (R \otimes T(s^{-1}\overline{C}) \otimes L, d_0 + d_1).$$

A generic element is denoted $r\langle c_1|c_2|\cdots|c_k\rangle l$ with degree $|r| + |l| + \sum_{1=1}^{k} |s^{-1}c_k]$. The differential d_0 is the unique derivation extending $-sd$, while d_1 is given by the formula

$$\begin{aligned} d_1(r\langle c_1|\cdots|c_{p-1}\rangle l) = &-\sum_k (-1)^{|r'_k|} r'_k\langle x''_k|c_1|\cdots|c_{p-1}\rangle l \\ &+ \sum_j (-1)^{\varepsilon_j} r\langle c_1|\cdots|c_{p-1}|y'_j\rangle l''_j \\ &+ \sum_{j=1}^{p-1}\sum_i (-1)^{\varepsilon_j + |c'_{ji}|} r\langle c_1|\cdots|c'_{ji}|c''_{ji}|\cdots|c_{p-1}\rangle l \end{aligned}$$

with $\varepsilon_j = |r| + |s^{-1}c_1| + \cdots + |s^{-1}c_{j-1}|$. Here $\Delta c_j = \sum_i c'_{ji} \otimes c''_{ji}$, $\Delta r = \sum_k r'_k \otimes x''_k$ and $\Delta l = \sum_j y'_j \otimes l''_j$ denote the non reduced (resp. reduced) diagonals, $c_j, c'_{ji}, c'_{ji}, x'_j, y''_i \in C$ (resp. $\overline{C}$), $l, l''_j \in L$ and $r, r'_i \in R$.

Hereafter we will use the *normalized* and the *non-counital* cobar constructions

$$\overline{\Omega}C = \overline{\Omega}(\Bbbk; C; \Bbbk) = (T(s^{-1}\overline{C}), \bar{d}), \text{ and } \widetilde{\Omega}C = \overline{\Omega}(C \oplus \Bbbk) = (T(s^{-1}C), \tilde{d}).$$

The next lemma arises then naturally from standard computations

Lemma 2.8. *Let C be an augmented differential graded chain coalgebra, $C = \Bbbk \oplus \overline{C}$. Suppose that $\overline{C} = \overline{C}_{\geqslant 1}$. Then,*

(a) $\overline{\Omega}C = (T(V), \bar{d})$ *is a quasi-free algebra,*
(b) $\widetilde{\Omega}C = (T(V \oplus \Bbbk\varepsilon), \tilde{d})$ *with $\tilde{d}\varepsilon = \varepsilon^2$ and $\tilde{d}v = \bar{d}v + \varepsilon v - (-1)^{|v|} v\varepsilon$, $v \in V$.*

3. Naturality of the Hochschild cohomology with respect to quasi-isomorphisms

Let $f : A \to B$ be a quasi-isomorphism of differential graded algebras. Then the two natural maps

$$\mathbf{C}^*(A;A) \xrightarrow{\mathbf{C}^*(A;f)} \mathbf{C}^*(A;B) \xleftarrow{\mathbf{C}^*(f;B)} \mathbf{C}^*(B;B)$$

are morphisms of differential graded algebras.

Lemma 3.1. *The maps* $\mathbf{C}^*(A;f)$ *and* $\mathbf{C}^*(f;B)$ *are quasi-isomorphisms.*

Proof. Since $B(A;A;A)$ is A^e-semifree, this follows from [6, Proposition 2.3]. □

The composition

$$HH^*(f) : HH^*(A,A) \xrightarrow{HH^*(A;f)} HH^*(A;B) \xrightarrow{(HH^*(f;B))^{-1}} HH^*(B;B)$$

is a then a natural linear isomorphism.

Lemma 3.2. *If* $f : A \to B$ *and* $g : B \to C$ *are quasi-isomorphisms of differential graded algebras, then* $HH_*(g \circ f) = HH_*(g) \circ HH_*(f)$.

Proof. This follows directly form the commutativity of the diagram

$$\begin{array}{ccccccccc}
\mathbf{C}^*(A;A) & \xrightarrow{\mathbf{C}^*(A;f)} & \mathbf{C}^*(A;B) & \xleftarrow{\mathbf{C}^*(f;B)} & \mathbf{C}^*(B;B) & \xrightarrow{\mathbf{C}^*(B;g)} & \mathbf{C}^*(B;C) & \xleftarrow{\mathbf{C}^*(g;C)} & \mathbf{C}^*(C;C) \\
\| & & \| & & & & \| & & \| \\
\mathbf{C}^*(A;A) & \xrightarrow{\mathbf{C}^*(A;f)} & \mathbf{C}^*(A;B) & \xrightarrow{\mathbf{C}^*(A;g)} & \mathbf{C}^*(A;C) & \xleftarrow{\mathbf{C}^*(f;C)} & \mathbf{C}^*(B;C) & \xleftarrow{\mathbf{C}^*(g;C)} & \mathbf{C}^*(C;C).
\end{array}$$ □

Proposition 3.3. $HH^*(f)$ *is an isomorphism of Gerstenhaber algebras.*

Proof. Since the maps

$$\mathbf{C}^*(A;A) \xrightarrow{\mathbf{C}^*(A;f)} \mathbf{C}^*(A;B) \xleftarrow{\mathbf{C}^*(f;A)} \mathbf{C}^*(B;B)$$

are quasi-isomorphisms of differential graded algebras, we have only to prove that $HH^*(f)$ is a morphism of Lie algebras. For a morphism of differential graded coalgebras $\psi : C \to C'$, we introduce the vector space $\mathrm{Coder}_\psi(C, C')$ of ψ-coderivations. Then Proposition 2.5 extends to give the following commutative diagram of complexes

$$\begin{array}{ccccc}
\mathbf{C}^*(A;A) & \xrightarrow{\mathbf{C}^*(A;f)} & \mathbf{C}^*(A;B) & \xleftarrow{\mathbf{C}^*(f;B)} & \mathbf{C}^*(B;B) \\
\beta_A \downarrow \cong & & \beta_{A,B} \downarrow \cong & & \beta_B \downarrow \cong \\
\mathrm{Coder}(\widetilde{\mathbb{B}}A) & \xrightarrow{f_1} & \mathrm{Coder}_{\widetilde{\mathbb{B}}(f)}(\widetilde{\mathbb{B}}A, \widetilde{\mathbb{B}}B)) & \xleftarrow{f_2} & \mathrm{Coder}(\widetilde{\mathbb{B}}B),
\end{array}$$

where f_1 and f_2 are the natural maps obtained by composition with *f*.

By [6, Proposition 3.1] the quasi-isomorphism *f* factors as the composite of two quasi-isomorphisms of differential graded algebras

$$A \underset{i}{\overset{\simeq}{\hookrightarrow}} A \coprod T(V) \underset{p}{\overset{\simeq}{\twoheadrightarrow}} B,$$

where $i: A \hookrightarrow A \coprod T(V)$ admits a linear retraction r, and $p: A \coprod T(V) \twoheadrightarrow B$ admits a linear section s. By Lemma 3.2, we have $HH^*(f) = HH^*(p) \circ HH^*(i)$. It suffices therefore to prove that $HH^*(p)$ and $HH^*(i)$ are morphisms of graded Lie algebras.

In the case $f = p$, $\mathbf{C}^*(A; f)$ admits the linear section $\mathrm{Hom}(TC(sA), r)$ and so f_1 is surjective. Let x_i, $i = 1, 2$, be cycles in $\mathrm{Coder}(\widetilde{\mathbb{B}}B)$. Since f_1 is a surjective quasi-isomorphism of complexes, there exists cycles y_i, in $\mathrm{Coder}\widetilde{\mathbb{B}}(A)$ such that

$$\widetilde{\mathbb{B}}(f) \circ y_i = f_1(y_i) = f_2(x_i) = x_i \circ \widetilde{\mathbb{B}}(f).$$

Thus

$$\begin{aligned} f_1([y_1, y_2]) &= \widetilde{\mathbb{B}}(f) \circ y_1 \circ y_2 - (-1)^{|y_1||y_2|}\widetilde{\mathbb{B}}(f) \circ y_2 \circ y_1 \\ &= x_1 \circ x_2 \circ \widetilde{\mathbb{B}}(f) - (-1)^{|x_1||x_2|} x_2 \circ x_1 \circ \widetilde{\mathbb{B}}(f) \\ &= f_2([x_1, x_2]) \end{aligned}$$

and if a_i (resp. b_i) denotes the class of x_i (resp. y_i), $i = 1, 2$ then $sHH^*(f)[b_1, b_2] = H(f_2)^{-1} \circ H(f_1)([b_1, b_2]) = [a_1, a_2]$.

In the case $f = i$, f_2 is surjective and the same argument works, mutatis mutandis. □

4. Hochschild cohomology and derivations

4.1. Let $\widetilde{\mathrm{Der}}\, A = \mathrm{Der}\, A \oplus sA$ and $\widetilde{A} = (T(V \oplus \Bbbk\varepsilon), \overline{d})$ be as defined in the introduction. The purpose of this section is the study of the maps

$$i_A : \widetilde{\mathrm{Der}}\, A \hookrightarrow \mathrm{Hom}(TC(sA), A) \quad \text{and} \quad j_A : \widetilde{\mathrm{Der}}\, A \to \mathrm{Der}\, \widetilde{A}.$$

The degree -1 linear map i_A is defined by putting

$$\begin{aligned} &i_A(\theta + sa)(1) = (-1)^{|a|+1}a && \text{if } a \in A, \\ &i_A(\theta + sa)([a_1]) = (-1)^{|\theta|}\theta(a_1), && \\ &i_A(\theta + sa)([a_1|\cdots|a_n]) = 0 && \text{if } n > 1. \end{aligned}$$

On the other hand, $j_A(\theta + sa)$ is the unique derivation of $\widetilde{A}$ defined by

$$\begin{aligned} &j_A(\theta + sa)(v) = \theta(v) \text{ for } v \in A \\ &j_A(\theta + sa)(\varepsilon) = (-1)^{|a|+1}a. \end{aligned}$$

A straightforward computation shows that $s \circ i_A$ and j_A are morphisms of complexes. We then have

Theorem 2. *Let $A = (TV, d)$ be a quasi-free algebra. Then $s \circ i_A$ and j_A are quasi-isomorphisms of differential graded Lie algebras.*

Proof. It follows from the definitions that $s \circ i_A$ and j_A are morphisms of differential graded Lie algebras. We prove in Lemmas 4.2 and 4.4 below that both are quasi-isomorphisms. □

We form the semifree A^e-module

$$R_A := (A \otimes (\Bbbk \oplus sV) \otimes A, \widetilde{\delta})$$

with $\widetilde{\delta}$ defined by

$$\begin{array}{lll} \widetilde{\delta}(1 \otimes \lambda \otimes 1) & = 0 & \text{if } \lambda \in \Bbbk, \\ \widetilde{\delta}(1 \otimes sv \otimes 1) & = v \otimes 1_{\Bbbk} \otimes 1 - S_V(dv) - 1 \otimes 1_{\Bbbk} \otimes v & \text{if } v \in V. \end{array}$$

Here S_V is the universal derivation of A-bimodules defined by

$$S_V : A \to A \otimes sV \otimes A,\ v_1 \ldots v_n \mapsto \sum_{i=1}^{n} (-1)^{|v_1 \cdots v_{i-1}|} v_1 \ldots v_{i-1} \otimes sv_i \otimes v_{i+1} \ldots v_n.$$

By [15, Theorem 1.4] the morphism $\Pi : \mathbb{B}(A; A; A) \twoheadrightarrow R_A$, defined by

$$\Pi([sa_1|\cdots|sa_n]) = \begin{cases} S_V(a_1) & \text{if } n = 1, \\ 0 & \text{if } n \neq 1 \end{cases}$$

is a quasi-isomorphism.

Lemma 4.2. *If A is a quasi-free algebra, $A = (TV, d)$, then $s \circ i_A : \widetilde{\mathrm{Der}}(A) \xrightarrow{\simeq} s\mathbf{C}^*(A; A)$ is a quasi-isomorphism.*

Proof. We consider the following diagram of complexes

$$\begin{array}{lcl} \mathrm{Hom}_{A^e}(R_A, A) & \underset{\cong}{\xrightarrow{\alpha_A}} & \widetilde{\mathrm{Der}}(A) \\ \mathrm{Hom}_{A^e}(\Pi, A) \downarrow & & \downarrow i_A \\ \mathrm{Hom}_{A^e}(\mathbb{B}(A; A; A), A) & = & \mathbf{C}^*(A; A), \end{array} \tag{4.3}$$

where α_A is the degree 1 isomorphism defined as follows:

- if $f \in \mathrm{Hom}(sV, A)$, then $\alpha_A(f) \in \mathrm{Der}\, A$ and $\alpha_A(f)(v) = (-1)^{|f|+1} f(sv)$,
- if $g \in \mathrm{Hom}(\Bbbk, A)$, then $\alpha_A(g) \in sA$ and $\alpha_A(g) = (-1)^{|g|+1} s(g(1_{\Bbbk}))$.

A straightforward computation shows that the diagram commutes. Since Π is a quasi-isomorphism between two A^e-semifree modules, $\mathrm{Hom}_{A^e}(\Pi, A)$ is a quasi-isomorphism. □

Lemma 4.4. *If A is a quasi-free algebra, $A = (TV, d)$, then $j_A : \widetilde{\mathrm{Der}}(A) \xrightarrow{\simeq} \mathrm{Der}(\widetilde{A})$ is a quasi-isomorphism of differential graded Lie algebras.*

Proof. Denote $\mathrm{Der}_0(\widetilde{A}) = \{\theta \in \mathrm{Der}(\widetilde{A}) | \theta(\varepsilon) = 0\}$ and $\mathrm{Der}_1(\widetilde{A}) = \mathrm{Der}(\widetilde{A})/\mathrm{Der}_0(\widetilde{A}) \cong \{\theta \in \mathrm{Der}(\widetilde{A}) | \theta(V) = 0\}$, with the quotient differential. We deduce from the definition of j_A a diagram of short exact sequences of complexes

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\mathrm{Der}(A) & \xrightarrow{j_0} & \mathrm{Der}_0(\widetilde{A}) \\
\downarrow & & \downarrow \\
\widetilde{\mathrm{Der}(A)} & \xrightarrow{j_A} & \mathrm{Der}(\widetilde{A}) \\
\downarrow & & \downarrow \\
sA & \xrightarrow{j_1} & \mathrm{Der}_1(\widetilde{A}) \\
\downarrow & & \downarrow \\
0 & & 0.
\end{array}$$

The differential $\tilde{d} = d' + \delta$ on $\widetilde{A}$ satisfies

$$d'(v) = d(v), \quad d'(\varepsilon) = -\varepsilon^2, \quad \delta(\alpha) = [\varepsilon, \alpha], \quad \text{for } \alpha \in V \oplus \Bbbk\varepsilon.$$

By killing the ideal I generated by ε we get a morphism $\pi\colon \widetilde{A} \to \widetilde{A}/I$ and a retraction $\mathrm{Der}_0(\widetilde{A}) \to \mathrm{Der}(A)$. This shows that $H(j_0)$ is injective.

The differential D on $\mathrm{Der}_0(\widetilde{A})$ satisfies $D(\theta) = [\tilde{d}, \theta] = [d', \theta]$. Remark that $(\widetilde{A}, d' = (TV, d) \coprod (T(\varepsilon), d'))$ is the direct sum of the two subcomplexes $(T(V), d)$ and (I, d'), and that (I, d') is acyclic. Therefore since $(T(V), d')$ is quasi-free, an induction on the degrees shows that each cycle θ in $\mathrm{Der}_0(\widetilde{A})$ is homologous to some cycle θ' with $\theta'(V) \subset T(V)$. This shows that $H(j_0)$ is surjective.

We now prove that $H(j_1)$ is also an isomorphism. This will imply the result by a five lemma argument. First of all, the morphism j_1 admits a retraction $\sigma : \mathrm{Der}_1(\widetilde{A}) \to sA$ defined by $\sigma(\theta) = (-1)^{|\theta|} s\pi\theta(\varepsilon)$, which implies that $H(j_1)$ is injective.

For $\theta \in \mathrm{Der}_1(\widetilde{A})$, we have $D(\theta) = d \circ \theta$. Once again, since I is acyclic for the differential d, each cycle $\theta \in \mathrm{Der}_1(\widetilde{A})$ is homologous to some θ' with $\theta'(\varepsilon) \subset T(V)$. This shows that $H(j_1)$ is surjective. □

5. The duality $\mathscr{D}_{\mathscr{C}}$ when C is a finite type coalgebra

In this section each chain coalgebra C satisfies the following properties:

- C is finite type,
- $C = \Bbbk \oplus \overline{C}$, with $\overline{C} = \overline{C}_{\geqslant 2}$.

To each map of coalgebras, $f : C \to C'$, we associate a morphism of complexes $\tilde{f} : \mathbf{C}^*(\overline{\Omega}C; \overline{\Omega}C') \to \mathbf{C}^*(C'^{\vee}; C^{\vee})$ defined by the composition'

$$\begin{array}{ccccc}
\mathbf{C}^*(\overline{\Omega}C; \overline{\Omega}C') & \cong & (\mathrm{Hom}(\widetilde{B}\overline{\Omega}C, \overline{\Omega}C'), D_0 + D_1) & \xrightarrow{\mathrm{Hom}(\sigma_C, \overline{\Omega}C')} & (\mathrm{Hom}(C, \overline{\Omega}C'), D_0 + D_1) \\
 & & & & \downarrow \mathrm{Hom}(C, i) \\
\mathbf{C}^*(C'^{\vee}; C^{\vee}) & \cong & (\mathrm{Hom}(\widetilde{B}(C')^{\vee}, C^{\vee}), D_0 + D_1) & \xleftarrow{\Gamma} & (\mathrm{Hom}(C, \widetilde{\Omega}C'), D_0 + D_1).
\end{array} \qquad (*)$$

Before going further we explain the differentials and morphisms involved in the definition of $\tilde{f}$. The differentials $D_0 + D_1$ on $\mathrm{Hom}(C, \overline{\Omega}C')$ and $\mathrm{Hom}(C, \widetilde{\Omega}C')$ are defined by

$$D_0(\varphi) = d \circ \varphi - (-1)^{|\varphi|}\varphi \circ d,$$
$$D_1(\varphi)(c) = -\left(\sum_i (-1)^{|c_i|\cdot|\varphi|}\langle f(c_i)\rangle \cdot \varphi(c_i')\right) + \sum_i (-1)^{|\varphi|+|c_i|}\varphi(c_i)\cdot\langle f(c_i')\rangle,$$

where $\bar{\Delta}_C(c) = \sum c_i \otimes c_i'$, and $\Delta_C(c) = \sum c_i \otimes c_i'$, respectively.

The morphism $i : \overline{\Omega}C \hookrightarrow \widetilde{\Omega}C$ is the canonical inclusion. The map $\sigma_C : C \xrightarrow{\simeq} \widetilde{\mathbb{B}}\overline{\Omega}C$ is the counity of the bar–cobar adjunction,

$$\sigma_C(c) = [\langle c\rangle] + \sum_{i\geqslant 1}\sum_j [\langle c_{1,j}\rangle|\cdots|\langle c_{i+1,j}\rangle],$$

where $\overline{\Delta}^i c = \sum_j c_{1,j} \otimes \ldots \otimes c_{i+1,j}$, $c \in \overline{C}$.

The map Γ is related to the usual duality process, $\Gamma(g) = g^\vee \circ \Theta$, where $\Theta : T(sC'^\vee) \to (T(s^{-1}C'))^\vee$ is defined by

$$\Theta([f_1|\cdots|f_n])(\langle c_1|\cdots|c_k\rangle) = \begin{cases} 0 & \text{if } k \neq n,\\ (-1)^n \varepsilon_\sigma f_1(c_1)\cdots f_n(c_n) & \text{if } k = n.\end{cases}$$

Here ε_σ is the graded sign obtained by the strict application of the Koszul rule sign to the graded permutation

$$sf_1, sf_2, \ldots, sf_n, s^{-1}c_1, s^{-1}c_2, \ldots, s^{-1}c_n \mapsto f_1, s, s^{-1}c_1, f_2, s, s^{-1}c_2, \ldots, f_n, s, s^{-1}c_n.$$

All the spaces occurring in the definition of $\tilde{f}$ are spaces of morphisms from a coalgebra into an algebra, they are therefore algebras and the diagram $(*)$ is a diagram of differential graded algebras

Proposition 5.1. *In the diagram* (*), *all the morphisms are quasi-isomorphisms of differential graded algebras. In particular,* $H_*(\tilde{f}) : HH_*(\overline{\Omega}C; \overline{\Omega}C') \to HH_*(C'^\vee; C^\vee)$ *and* $\mathscr{D}_C = H_*(\widetilde{id_C})$ *are isomorphisms of graded algebras.*

Proof. The morphism $\overline{\Omega}C \otimes \sigma_C \otimes \overline{\Omega}C$ is a homomorphism of differential graded modules [11, p. 209] and we have the following commutative diagram of complexes:

$$\begin{array}{ccc} (\mathrm{Hom}(C, \overline{\Omega}C'), D_0 + D_1) & \overset{\Phi}{\underset{\cong}{\longleftarrow}} & \mathrm{Hom}_{\overline{\Omega}C^e}(R_{\overline{\Omega}C}, \overline{\Omega}C') \\ \mathrm{Hom}(\sigma_C, \overline{\Omega}C') \uparrow & & \uparrow \mathrm{Hom}_{\overline{\Omega}C^e}(1 \otimes \sigma_C \otimes 1, \overline{\Omega}C') \\ (\mathrm{Hom}(\widetilde{\mathbb{B}}\overline{\Omega}C, \overline{\Omega}C'), D_0 + D_1) & \overset{\Phi}{\underset{\cong}{\longleftarrow}} & \mathrm{Hom}_{\overline{\Omega}C^e}(\mathbb{B}(\overline{\Omega}C; \overline{\Omega}C; \overline{\Omega}C), \overline{\Omega}C'), \end{array}$$

where $R_{\overline{\Omega}C}$ is defined as in Section 4 and where the horizontal lines are defined by restriction. Since the right vertical map is a quasi-isomorphism and since the horizontal maps are isomorphisms, $\mathrm{Hom}(\sigma_C, \overline{\Omega}C')$ is also a quasi-isomorphism.

The inclusion $i : \overline{\Omega}C' \hookrightarrow \widetilde{\Omega}C'$ is a quasi-isomorphism of coalgebras and the induced map $\mathrm{Hom}(C, i)$ is therefore a quasi-isomorphism of differential graded algebras.

By the finite type hypothesis on C, the morphism $\Theta : T(sC'^{\vee}) \to (T(s^{-1}C'))^{\vee}$ is an isomorphism of coalgebras. Lemma 5.2 below permits to conclude. □

Lemma 5.2. *Γ is an isomorphism of complexes.*

Proof. Let M and N be finite type complexes, and $\Gamma : \mathrm{Hom}(M, N) \to \mathrm{Hom}(N^{\vee}, M^{\vee})$ be the dualization map. Then

$$\begin{aligned}(Df^{\vee})(\varphi) &= D(f^{\vee}(\varphi)) - (-1)^{|f|} f^{\vee}(D\varphi)\\ &= (-1)^{|f|\cdot|\varphi|+|f|+|\varphi|+1}\varphi \circ f \circ d - (-1)^{|\varphi|+|f|\cdot|\varphi|+1}\varphi \circ d \circ f\\ &= (-1)^{|\varphi|+|f|\cdot|\varphi|}\varphi(Df) = (Df)^{\vee}(\varphi).\end{aligned}$$

This implies in our case that $D_0\Gamma = \Gamma D_0$. Concerning D_1, a standard computation shows that $(D_1\varphi^{\vee})[a_1|\cdots|a_k](c)$ and $(D_1\varphi)^{\vee}[a_1|\cdots|a_k](c)$ are both equal to

$$\begin{aligned}&\sum_i (-1)^{|\varphi|\cdot(\sum|sa_j|)+|c_i|(|\varphi|+|sa_2|+\cdots+|sa_k|)} a_1(f(c_i))\cdot[a_2|\cdots|a_k](\varphi(c_i'))\\ &\quad+\sum_i (-1)^{|\varphi|+(|\varphi|+1)(|sa_1|+\cdots+|a_{k-1}|)+|c_i|\cdot|a_k|}[a_1|\cdots|a_{k-1}](\varphi(c_i))\cdot a_k(f(c_i')).\end{aligned}$$

This ends the proof of the lemma. □

Proposition 5.3. *The morphism $\mathscr{D}_C : HH_*(\overline{\Omega}C; \overline{\Omega}C) \to HH_*(C^{\vee}; C^{\vee})$ is a morphism of graded Lie algebras.*

Proof. We use the quasi-isomorphism

$$\Pi : \mathbb{B}(\overline{\Omega}C; \overline{\Omega}C; \overline{\Omega}C) \to R_{\overline{\Omega}C}$$

of semifree modules described in Section 4 with $A = \overline{\Omega}C$. The morphism $\overline{\Omega}C \otimes \sigma_C \otimes \overline{\Omega}C$ is a section of Π, and therefore, denoting $\pi = \Bbbk \otimes \Pi \otimes \Bbbk$, the morphism $H^*(\mathrm{Hom}(\pi, \overline{\Omega}C))$ is an isomorphism inverse to $H^*(\mathrm{Hom}(\sigma_C, \overline{\Omega}C))$.

The proof follows then from considerations about the following diagram

$$\begin{array}{ccc} s\,\mathrm{Hom}(\widetilde{B}\overline{\Omega}C, \overline{\Omega}C) & \xleftarrow{i_{\overline{\Omega}C}} & \widetilde{\mathrm{Der}}(\overline{\Omega}C)\\ \uparrow s\,\mathrm{Hom}(\pi, \overline{\Omega}C) & & \|\\ s\,\mathrm{Hom}(C, \overline{\Omega}C) & \xrightarrow{a_{\overline{\Omega}C}} & \widetilde{\mathrm{Der}}(\overline{\Omega}C)\\ \downarrow s\,\mathrm{Hom}(C, i) & & \downarrow j_{\overline{\Omega}C}\\ s\,\mathrm{Hom}(C, \widetilde{\Omega}C) & \xrightarrow{\varphi} & \mathrm{Der}(\widetilde{\Omega}C)\\ \downarrow s\Gamma & & \downarrow \Gamma\\ s\,\mathrm{Hom}(\widetilde{B}(C)^{\vee}, C^{\vee}) & \overset{\cong}{\to} & \mathrm{Coder}(\widetilde{B}C^{\vee}).\end{array}$$

The commutativity of the upper square is proved in (4.4), the commutativity of the middle square follows from the definitions of the morphisms, and the lower square commutes trivially. The right-hand side maps are morphisms of differential graded Lie algebras and the upper and lower horizontal maps are the maps that define the Lie algebra structures on the Hochschild complexes. □

Proposition 5.4. *The morphism $\mathscr{D}_C$ is natural with respect to quasi-isomorphism of coalgebras, i.e. if $f : C \to C'$ is a quasi-isomorphism of differential graded coalgebras, then we have a commutative diagram of isomorphic Gerstenhaber algebras*

$$\begin{array}{ccc} HH_*(\overline{\Omega}C;\overline{\Omega}C) & \xrightarrow{HH^*(\overline{\Omega}f)} & HH_*(\overline{\Omega}C';\overline{\Omega}C') \\ \mathscr{D}_C \downarrow & & \downarrow \mathscr{D}_{C'} \\ HH_*(C^\vee;C^\vee) & \xleftarrow{HH^*(f^\vee)} & HH^*(C'^\vee;C'^\vee). \end{array}$$

Proof. This follows directly from the commutativity of the diagram

$$\begin{array}{ccccc} \mathrm{Hom}(\widetilde{B}\overline{\Omega}C,\overline{\Omega}C) & \xrightarrow{\mathrm{Hom}(\widetilde{B}\overline{\Omega}C,\overline{\Omega}f)} & \mathrm{Hom}(\widetilde{B}\overline{\Omega}C,\overline{\Omega}C') & \xleftarrow{\mathrm{Hom}(\widetilde{B}\overline{\Omega}f,\overline{\Omega}C')} & \mathrm{Hom}(\widetilde{B}\overline{\Omega}C',\overline{\Omega}C') \\ \downarrow \mathrm{Hom}(\sigma_C,\overline{\Omega}C) & & \downarrow \mathrm{Hom}(\sigma_C,\overline{\Omega}C') & & \downarrow \mathrm{Hom}(\sigma_{C'},\overline{\Omega}C') \\ \mathrm{Hom}(C,\overline{\Omega}C) & \xrightarrow{\mathrm{Hom}(C,\overline{\Omega}f)} & \mathrm{Hom}(C,\overline{\Omega}C') & \xleftarrow{\mathrm{Hom}(f,\overline{\Omega}C')} & \mathrm{Hom}(C',\overline{\Omega}C') \\ \downarrow \mathrm{Hom}(C,i) & & \downarrow \mathrm{Hom}(C,i) & & \downarrow \mathrm{Hom}(C',i) \\ \mathrm{Hom}(C,\widetilde{\Omega}C) & \xrightarrow{\mathrm{Hom}(C,\widetilde{\Omega}f)} & \mathrm{Hom}(C,\widetilde{\Omega}C') & \xleftarrow{\mathrm{Hom}(f,\widetilde{\Omega}C')} & \mathrm{Hom}(C',\widetilde{\Omega}C') \\ \downarrow \Gamma & & \downarrow \Gamma & & \downarrow \Gamma \\ \mathrm{Hom}(\widetilde{B}(C)^\vee,C^\vee) & \xrightarrow{\mathrm{Hom}(\widetilde{B}(f)^\vee,C^\vee)} & \mathrm{Hom}(\widetilde{B}(C')^\vee,C^\vee) & \xleftarrow{\mathrm{Hom}(\widetilde{B}(C')^\vee,f^\vee)} & \mathrm{Hom}(\widetilde{B}(C')^\vee,(C')^\vee). \end{array}$$

The upper horizontal line induces in homology the isomorphism $HH^*(\overline{\Omega}f)$ and the lower horizontal line the isomorphism $HH^*(f^\vee)$. □

6. The duality map $\mathscr{D}_{\mathscr{C}}$ when $H(C)$ is of finite type

Let C be a differential graded coalgebra satisfying to the hypothesis of the introduction: C is supplemented, $C = \Bbbk \oplus \overline{C}$, $H(C)$ is finitely generated, and $H(\overline{C}) = H(\overline{C})_{\geqslant 2}$.

The dual differential graded algebra $C^\vee$ admits therefore a minimal model $(T(V), d)$ in the sense of Halperin-Lemaire [9]. This means that there is a quasi-isomorphism $f : (T(V), d) \to C^\vee$ and that $d(V) \subset T^{\geqslant 2}(V)$. In particular $T(V)$ is of finite type. Denote by $e : C \to (C^\vee)^\vee$ the canonical inclusion of C in its bidual, then the composition $\varphi : C \xrightarrow{e} (C^\vee)^\vee \xrightarrow{f^\vee} (T(V))^\vee$ is a quasi-isomorphism of differential graded coalgebras.

We define $\mathscr{D}_C$ to be the composite

$$\begin{array}{ccc} HH^*(\overline{\Omega}C;\overline{\Omega}C) & \xrightarrow{HH^*(\overline{\Omega}\varphi)} & HH^*(\overline{\Omega}(TV)^\vee;\overline{\Omega}(TV)^\vee) \\ & & \mathscr{D}_{(TV)^\vee} \downarrow \\ & & HH^*(TV;TV) \xrightarrow{HH^*(\varphi^\vee)} HH^*(C^\vee;C^\vee). \end{array}$$

Proposition 6.1. *The isomorphism $\mathscr{D}_C$ does not depend on the choice of f. Moreover, associated to each quasi-isomorphism of differential graded coalgebras $g : C \to C'$ there* *is a commutative diagram*

$$\begin{array}{ccc} HH_*(\overline{\Omega}C;\overline{\Omega}C) & \xrightarrow{HH^*(\overline{\Omega}g)} & HH_*(\overline{\Omega}C';\overline{\Omega}C') \\ \mathscr{D}_C\downarrow & & \downarrow\mathscr{D}_{C'} \\ HH_*(C^\vee;C^\vee) & \xleftarrow{HH^*(g^\vee)} & HH^*(C'^\vee;C'^\vee). \end{array}$$

Proof. Let $f:(T(V),d)\to C^\vee$ and $f':(T(V),d)\to C^\vee$ be two quasi-isomorphisms. Then there exists a finite type differential graded algebra (E,d), a quasi-isomorphism $h:(E,d)\to C^\vee$ and factorizations of *f* and f' through *h*: $f=h\circ k$, $f'=h\circ l$.

$$\begin{array}{ccc} (T(V),d) & & \\ k\downarrow & \overset{f}{\searrow} & \\ (E,d) & \overset{h}{\to} & C^\vee. \\ l\uparrow & \overset{f'}{\nearrow} & \\ (T(V),d) & & \end{array}$$

Using Proposition 5.4, we obtain then a commutative diagram

$$\begin{array}{ccccc} HH^*(\overline{\Omega}C;\overline{\Omega}C) & = & HH^*(\overline{\Omega}C;\overline{\Omega}C) & = & HH^*(\overline{\Omega}C;\overline{\Omega}C) \\ HH^*(\overline{\Omega}f^\vee)\downarrow & & HH^*(\overline{\Omega}h^\vee)\downarrow & & HH^*(\overline{\Omega}f'^\vee)\downarrow \\ HH^*(\overline{\Omega}(TV);\overline{\Omega}(TV)) & \xleftarrow{HH^*(\overline{\Omega}k^\vee)} & HH^*(\overline{\Omega}E;\overline{\Omega}E) & \xrightarrow{HH^*(\overline{\Omega}l^\vee)} & HH^*(\overline{\Omega}(TV);\overline{\Omega}(TV)) \\ \mathscr{D}_{(TV)^\vee}\downarrow & & \mathscr{D}_{E^\vee}\downarrow & & \mathscr{D}_{(TV)^\vee}\downarrow \\ HH^*(TV,TV) & \xrightarrow{HH^*(k)} & HH^*(E,E) & \xleftarrow{HH^*(l)} & HH^*(TV;TV) \\ HH^*(f)\downarrow & & HH^*(h)\downarrow & & HH^*(f')\downarrow \\ HH^*(C^\vee;C^\vee) & = & HH^*(C^\vee;C^\vee) & = & HH^*(C^\vee;C^\vee). \end{array}$$

This implies directly the independency of $\mathscr{D}_C$ with respect to *f*.

Let now $g:C\to C'$ be a quasi-isomorphism of differential graded coalgebras. We choose a minimal model $f:(T(V),d)\to(C')^\vee$, and we deduce that a minimal model for $C^\vee$ is the composition $g^\vee\circ f$. Therefore, $\mathscr{D}_C$ is the composition

$$\begin{array}{ccccc} HH^*(\overline{\Omega}C;\overline{\Omega C}) & \overset{HH^*(\overline{\Omega}g)}{\to} & HH^*(\overline{\Omega}C';\overline{\Omega}C') & \overset{HH^*(\overline{\Omega}f^\vee)}{\to} & HH(\overline{\Omega}(TV)^\vee;\overline{\Omega}(TV)^\vee)) \\ & & & & \downarrow\mathscr{D}_{(TV)^\vee} \\ HH^*(C^\vee;C^\vee) & \overset{HH^*(g^\vee)}{\leftarrow} & HH^*((C')^\vee;(C')^\vee) & \overset{HH^*(f)}{\leftarrow} & HH^*(TV;TV). \end{array}$$

The proposition follows then from the definitions of $\mathscr{D}_C$ and $\mathscr{D}_{C'}$. □

7. Examples

Let $(TV,d)\to C_*(\Omega M)$ be an Adams–Hilton model of *M*, [1]. The fundamental result of Adams, Theorems 1 and 2 furnish together the isomorphisms of graded Lie algebras

$$sHH^*(\mathscr{C}^*M;\mathscr{C}^*M)\cong sHH^*(\mathscr{C}_*(\Omega M);\mathscr{C}_*(\Omega M))\cong H_*(\widetilde{\mathrm{Der}}(T(x))).$$

We apply those results to compute explicitly the Lie algebra structure when $M=S^n$ and $M=\mathbb{C}P^2$.

Example 1. $M = S^r$.

The Adams–Hilton model is $(T(x), 0)$ with $|x| = r - 1$.

A linear basis of $\widetilde{\mathrm{Der}}(T(x))$ is given by the elements $\theta_n \in \mathrm{Der}(T(x))$, $n \geqslant -1$, $\theta_n(x) = x^{n+1}$, and sx^n, $n \geqslant 0$. By construction $|\theta_n| = (r-1)n$ and $|sx^n| = (r-1)n + 1$, $D(\theta_n) = 0$ and

$$D(sx^n) = \begin{cases} 0 & \text{if } (r-1)n \text{ is even,} \\ -2\theta_n & \text{if } (r-1)n \text{ is odd.} \end{cases}.$$

When r is even and $\mathbb{k}$ is a field of characteristic different from 2, a linear basis of $H_*(\widetilde{\mathrm{Der}}(T(x))$ is thus given by the elements

$$\theta_{-1}, \theta_0, s1, \theta_2, sx^2, \theta_3, \ldots.$$

When r is odd or when $\mathbb{k}$ is a field of characteristic 2, then the differential is zero and a linear basis for the homology is given by the elements θ_n and sx^n.

When r is even and if the characteristic is different from 2, the elements θ_{-1} and $s1$ are central elements, and the other brackets satisfy

$$[\theta_p, \theta_q] = (q-p)\theta_{p+q}, \quad [\theta_p, sx^q] = qsx^{q+p} \quad [sx^r, sx^s] = 0.$$

When r is odd or when the characteristic is two, the bracket is defined by a similar formula, with $p, q \geqslant -1$

$$[\theta_p, \theta_q] = (q-p)\theta_{p+q}, \quad [\theta_p, sx^q] = qsx^{q+p} \quad [sx^r, sx^s] = 0.$$

In both cases, the graded Lie algebra $H_*(\widetilde{\mathrm{Der}}(T(x)))$ is generated by elements of degrees $\leqslant 4$.

Example 2. Let $M = \mathbb{C}P^2$ and $\mathbb{k} = \mathbb{Q}$. An Adams–Hilton model for M is given by the quasi-isomorphism

$$(T(x, y), d) \to \mathscr{C}_*(\Omega M), \quad \mathrm{d}x = 0, \mathrm{d}y = x^2, |x| = 1, |y| = 3.$$

The homology of $(T(x, y), d)$ is $\wedge x \otimes \mathbb{Q}[t]$ with $t = xy + yx$. A linear basis of $H_*(\mathrm{Der}T(x, y))$ is given by the classes of the cycles φ_n, ψ_n, ρ_n and σ_n, $n \geqslant 0$.

$$\begin{array}{ll} \varphi_n(x) = t^n, \varphi_n(y) = a_n, & |\varphi_n| = 4n - 1, \\ \psi_n(x) = xt^n, \psi_n(y) = xa_n + 2yt^n, & |\psi_n| = 4n, \\ \rho_n(x) = 0, \rho_n(y) = t^n, & |\rho_n| = 4n - 3, \\ \sigma_n(x) = 0, \sigma_n(y) = xt^n, & |\sigma_n| = 4n - 2, \end{array}$$

where $a_0 = 0$ and for $n > 0$, $a_n = t^{n-1}y^2 + t^{n-2}y^2t + \cdots + y^2t^{n-1}$.

On the other hand, a linear basis for $H_*(sT(x, y))$ is given by the classes of the elements st^n and sxt^n.

Now by standard computations we have

Proposition 7.1. *A linear basis of $H_*(\widetilde{\mathrm{Der}}(T(x,y))$ is given by the classes of the elements*

$$\varphi_n, \psi_n, \sigma_n, st^n, n \geqslant 0, \text{ and } \rho_0.$$

There is exactly one element by degree

Degree:	−3	−2	−1	0	1	2	3	4	5	6	7	8	9	10
Element:	ρ_0	σ_0	φ_0	ψ_0	$s1$	σ_1	φ_1	ψ_1	st	σ_2	φ_2	ψ_2	st^2	σ_3

The element ρ_0 is central and the Lie bracket in $HH^*(\mathscr{C}^*M;\mathscr{C}^*M)$ satisfies

$$\begin{array}{ll}
[\varphi_n,\psi_m] = (1-3n)\varphi_{n+m}, & [\varphi_n,\varphi_m] = 0, \\
[\psi_n,\psi_m] = 3(m-n)\psi_{n+m}, & [\varphi_n,\sigma_m] = 0, \\
[\sigma_n,\sigma_m] = 0, & [\psi_r,\sigma_n] = (3(n-r)-1)\sigma_{n+r}, \\
[st^n, st^m] = 0, & [st^n,\varphi_m] = 0, \\
[st^n,\sigma_m] = 0, & [\psi_n, st^m] = 3mst^{n+m}.
\end{array}$$

In particular the Lie algebra $HH^*(\mathscr{C}^*M;\mathscr{C}^*M)$ is generated by the elements of degrees $\leqslant 8$.

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