BATALIN-VILKOVISKY ALGEBRAS AND CYCLIC COHOMOLOGY OF HOPF ALGEBRAS

LUC MENICHI

Abstract. We show that the Connes-Moscovici negative cyclic cohomology of a Hopf algebra equipped with a character has a Lie bracket of degree -2. More generally, we show that a "cyclic operad with multiplication" is a cocyclic module whose simplicial cohomology is a Batalin-Vilkovisky algebra and whose negative cyclic cohomology is a graded Lie algebra of degree -2. This generalizes the fact that the Hochschild cohomology algebra of a symmetric algebra is a Batalin-Vilkovisky algebra.

1. Introduction

Let $\mathbb{k}$ be an arbitrary commutative ring and denote by $\mathcal{H}$ an (ungraded) bialgebra over $\mathbb{k}$. We denote by $\Omega\mathcal{H}$ the Adams Cobar construction on $\mathcal{H}$. Its cohomology is $\mathrm{Cotor}^*_{\mathcal{H}}(\mathbb{k},\mathbb{k})$. It results from [7, p. 65] that $\mathrm{Cotor}^*_{\mathcal{H}}(\mathbb{k},\mathbb{k})$ has a Gerstenhaber algebra structure.

On the other hand, assume that $\mathcal{H}$ has an involutive antipode or more generally that $\mathcal{H}$ is a Hopf algebra equipped with a modular pair in involution where the group like element is the unit 1 of $\mathcal{H}$. Connes and Moscovici [3, 4] have proved that $\Omega\mathcal{H}$ has a canonical cocyclic module structure. A cocyclic module gives a cochain complex equipped with a Connes coboundary map B and therefore in cohomology an operator B. Since a Batalin-Vilkovisky algebra (Definition 3.1) is a Gerstenhaber algebra equipped with an operator B, it is natural to conjecture that $\mathrm{Cotor}^*_{\mathcal{H}}(\mathbb{k},\mathbb{k})$ is a Batalin-Vilkovisky algebra. The first result of this paper is to prove that conjecture.

Theorem 1.1. *Let $\mathcal{H}$ be a Hopf algebra endowed with a modular pair in involution $(\chi, 1)$. Then the canonical algebra structure of the Cobar construction on $\mathcal{H}$ together with its Connes-Moscovici cocyclic structure, define a Batalin-Vilkovisky algebra structure on $Cotor^*_{\mathcal{H}}(\mathbb{k},\mathbb{k})$.*

A cocyclic module gives a non-positively lower graded mixed complex. We call *negative cyclic cohomology* of the cocyclic module, the cyclic homology of the associated mixed complex [11, 2.5.13 without the hypothesis C non-negatively graded]. We obtain

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Corollary 1.2. *The negative cyclic cohomology of* $\mathcal{H}$*, denoted* $HC^*_{-(\chi,1)}(\mathcal{H})$*, is a graded Lie algebra of degree* -2*.*

The easiest way to see that the cotorsion product of a bialgebra $\mathcal{H}$ is a Gerstenhaber algebra, is to remark as in [7] that the Cobar construction on $\mathcal{H}$, $\Omega\mathcal{H}$, is an operad with multiplication (Definition 2.4) and to apply the following general theorem.

1.3. [7, 8, 13] *a) Each operad with multiplication* O *is a cosimplicial module (See 2.5). Denote by* $\mathcal{C}^*(O)$ *the associated cochain complex.*
b) Its cohomology $H^*(\mathcal{C}^*(O))$ *is a Gerstenhaber algebra.*

To prove Theorem 1.1, we proceed similarly:
-we introduce the notion of cyclic operad with multiplication (Definition 3.11),
-we show in section 5 that $\Omega\mathcal{H}$ is a cyclic operad with multiplication.
-we prove the main result of this paper.

Theorem 1.4. *If* O *is a cyclic operad with a multiplication then*
a) the structure of cosimplicial module on O *extends to a structure of cocyclic module and*
b) the Connes coboundary map B *on* $\mathcal{C}^*(O)$ *induces a natural structure of Batalin-Vilkovisky algebra on the Gerstenhaber algebra* $H^*(\mathcal{C}^*(O))$*.*

Corollary 1.5. *The negative cyclic cohomology of* $\mathcal{C}^*(O)$*,* $HC^*_-(\mathcal{C}^*(O))$*, has naturally a graded Lie algebra structure of degree* -2*.*

Theorem 1.4 is inspired by a result (See section 8) announced by McClure and Smith in [14].

In representation theory [5], an algebra A is *symmetric* if A is equipped with an isomorphism of A-bimodules $\Theta : A \overset{\cong}{\to} A^\vee$ between A and its dual $\text{Hom}(A,\mathbb{k})$. As a second application of Theorem 1.4, we show

Theorem 1.6. *Let* A *be a symmetric algebra. Then the Connes coboundary map on* $HH^*(A,A^\vee)$ *defines via the isomorphism* $HH^*(A,\Theta) : HH^*(A,A) \overset{\cong}{\to} HH^*(A,A^\vee)$ *a structure of Batalin-Vilkovisky algebra on the Gerstenhaber algebra* $HH^*(A,A)$*.*

Corollary 1.7. *The negative cyclic cohomology of* A*,* $HC^*_-(A)$*, is a graded Lie algebra of degree* -2*.*

Remark that when $\mathbb{k}$ is a field, if A is finite dimensional over $\mathbb{k}$ then $HC^*_-(A)$ is the dual of the negative cyclic homology $HC^-_*(A)$.

Theorem 1.6 has been proved by Tradler [18]. In fact, he proved Theorem 1.6 much more generally, for "homotopy" symmetric algebras. Our proof for "strict" symmetric algebras is much simpler.

Tamarkin and Tsygan [17, Conjecture 0.13] have conjectured a related result at the chain level. See also McClure and Smith [14, section 16.2].

Moreover Tamarkin and Tsygan have mentionned a relation between Theorem 1.6 and Connes-Moscovici cyclic cohomology of Hopf algebras [17]. Theorem 1.4 establishes such relation.

The main tools in the proof of Theorem 1.4 are the following results which have their own interest. Let $\overline{\mathcal{C}}^*(O)$ be the normalized cochain complex associated to the cyclic operad with multiplication O and B the Connes normalized coboundary map on $\overline{\mathcal{C}}^*(O)$. Denote by $\cup$ the cup product and by $\overline{\circ}$ the composition product in $\overline{\mathcal{C}}^*(O)$ (See (2.6) and (2.7)).

Lemma 1.8. *There is a bilinear map Z (See (6.1)) of degree -1 such that*

$$B(f\cup g)=Z(f,g)+(-1)^{mn}Z(g,f),\quad \forall f\in\overline{\mathcal{C}}^m(O), g\in\overline{\mathcal{C}}^n(O).$$

Proposition 1.9. *There is a bilinear map H (See (6.4)) of degree -2 such that, for any $f\in\overline{\mathcal{C}}^m(O)$ and $g\in\overline{\mathcal{C}}^n(O)$,*

$$(-1)^m\left(Z(f,g)-(Bf)\cup g\right)-f\overline{\circ}g$$
$$=dH(f,g)+H(df,g)+(-1)^{m-1}H(f,dg).$$

Up to the signs, the second member of this Proposition is simply the operator $[d,H]$ applied to $f\otimes g$.

We give now the plan of the paper:

2) operads with multiplication. This section is a review on operads with multiplication. We recall the definition of operad with multiplication. We define the structure of Gerstenhaber algebra associated to an operad with multiplication. We recall the two fundamental examples of operad with multiplication:

-the endomorphism operad of an algebra,

-the Cobar construction on a bialgebra.

3) cyclic operad with multiplication. We introduce the notions of cyclic operad and cyclic operad with multiplication. We prove part a) of Theorem 1.4.

4) Hochschild cohomology of a symmetric algebra. We prove Theorem 1.6 by showing that the endomorphism operad of a symmetric algebra is a cyclic operad with multiplication.

5) Cyclic cohomology of Hopf algebras. We recall what a Hopf algebra $\mathcal{H}$ endowed with a modular pair in involution of the form $(\chi,1)$ is and we prove Theorem 1.1 by showing that the Cobar construction on $\mathcal{H}$ is a cyclic operad with multiplication.

6) Proof of part b) of Theorem 1.4. We prove Lemma 1.8. Then we deduce part b) of Theorem 1.4 from Lemma 1.8 and Proposition 1.9. Finally, we prove Proposition 1.9.

7) Proof of Corollary 1.5. We define the Lie bracket on negative cyclic cohomology in the same way as Chas and Sullivan define a Lie bracket on S^1-equivariant homology in [2].

8) Comparison with McClure and Smith. We compare Theorem 1.4 with two results announced by McClure and Smith in [14].

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2. Operads with multiplication

2.1. A *Gerstenhaber algebra* is a graded module $G = \{G^i\}_{i\in\mathbb{Z}}$ equipped with two linear maps

$$\cup : G^i \otimes G^j \to G^{i+j}\,, \quad x \otimes y \mapsto x \cup y$$
$$\{-,-\} : G^i \otimes G^j \to G^{i+j-1}\,, \quad x \otimes y \mapsto \{x, y\}$$

such that:
a) the cup product $\cup$ makes G into a graded commutative algebra
b) the bracket $\{-,-\}$ gives G a structure of graded Lie algebra of degree -1. This means that for each a, b and $c \in G$
$\{a, b\} = -(-1)^{(|a|-1)(|b|-1)}\{b, a\}$ and
$\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{(|a|-1)(|b|-1)}\{b, \{a, c\}\}$.
c) the cup product and the Lie bracket satisfy the Poisson rule. This means that for any $c \in G^k$ the adjunction map $\{-, c\} : G^i \to G^{i+k-1}, \quad a \mapsto \{a, c\}$ is a $(k-1)$-derivation: ie. for a, b, $c \in G$, $\{ab, c\} = \{a, c\}b + (-1)^{|a|(|c|-1)}a\{b, c\}$.

Usually, this definition is given for a lower graded module $G = \{G_i\}_{i\in\mathbb{Z}}$. If you put $G_i = G^{-i}$ as usual, you pass from an upper degree graded module to a lower graded module and the Lie bracket is of the usual (lower) degree $+1$.

2.2. In this paper, *operad* means non-Σ-operad in the category of $\Bbbk$-modules. That is: a sequence of modules $\{O(n)\}_{n\geq 0}$, an identity element $id \in O(1)$ and structure maps
$\gamma : O(n) \otimes O(i_1) \otimes \cdots \otimes O(i_n) \to O(i_1 + \cdots + i_n)$
$f \otimes g_1 \otimes \cdots \otimes g_n \mapsto \gamma(f; g_1, \ldots, g_n)$
satisfying associativity and unit [12].

Hereafter we use mainly the composition operations $\circ_i : O(m) \otimes O(n) \to O(m+n-1)$ $f \otimes g \mapsto f \circ_i g$ defined for $m \in \mathbb{N}^*$, $n \in \mathbb{N}$ and $1 \leq i \leq m$ by $f \circ_i g := \gamma(f; id, \ldots, g, id, \ldots, id)$ where g is the i-th element after the semicolon.

Example 2.3. [12] Let V be a module. The *endomorphism operad* of V is the operad $\mathcal{E}nd_V$ defined by $\mathcal{E}nd_V(n) := \text{Hom}(V^{\otimes n}, V)$. The identity element of $\mathcal{E}nd_V$ is the identity map $id_V : V \to V$.

2.4. An *operad with multiplication* is an operad equipped with an element $\mu \in O(2)$ called the multiplication and an element $e \in O(0)$ such that $\mu \circ_1 \mu = \mu \circ_2 \mu$ and $\mu \circ_1 e = id = \mu \circ_2 e$.

In [7], an operad with multiplication is called a strict unital comp algebra.

Let $\underline{Ass}$ be the (non-Σ) associative operad [12]: $\underline{Ass}(n) := \Bbbk$. An operad O is an operad with multiplication if and only if O is equipped with a morphism of operads $\underline{Ass} \to O$.

Sketch of proof of 1.3. a) The coface maps $\delta_i : O(n) \to O(n+1)$ and codegeneracy maps $\sigma_i : O(n) \to O(n-1)$ are defined [13] by

(2.5) $\delta_0 f = \mu \circ_2 f,\ \delta_i f = f \circ_i \mu,\ \delta_{n+1} f = \mu \circ_1 f,\ \sigma_{i-1} f = f \circ_i e$ for $1 \leq i \leq n$.

b) The associated cochain complex $\mathcal{C}^*(O)$ is the cochain complex whose differential d is given by

$$d := \sum_{i=0}^{n+1} (-1)^i \delta_i : O(n) \to O(n+1).$$

The linear maps $\cup : O(m) \otimes O(n) \to O(m+n)$ defined by

$$f \cup g := (\mu \circ_1 f) \circ_{m+1} g = (\mu \circ_2 g) \circ_1 f \tag{2.6}$$

gives $\mathcal{C}^*(O)$ a structure of differential graded algebra. The linear maps of degree -1

$$\overline{\circ}, \{-,-\} : O(m) \otimes O(n) \to O(m+n-1)$$

are defined by

$$f \overline{\circ} g := (-1)^{(m-1)(n-1)} \sum_{i=1}^{m} (-1)^{(n-1)(i-1)} f \circ_i g \tag{2.7}$$

and

$$\{f, g\} := f \overline{\circ} g - (-1)^{(m-1)(n-1)} g \overline{\circ} f.$$

The bracket $\{-,-\}$ defines a structure of differential graded Lie algebra of degree -1 on $\mathcal{C}^*(O)$. After passing to cohomology, the cup product $\cup$ and the bracket $\{-,-\}$ satisfy the Poisson rule. □

For further use, we prove with some details the following two corollaries.

Corollary 2.8. [6] *The Hochschild cohomology of an algebra, $HH^*(A,A)$, is a Gerstenhaber algebra.*

Proof. Let A be an associative algebra with multiplication $\mu : A \otimes A \to A$ and unit $e : \mathbb{k} \to A$. Then the endomorphism operad $\mathcal{E}nd_A$ of A equipped with μ and e is an operad with multiplication. The Hochschild cochain complex of A, denoted $\mathcal{C}^*(A,A)$, is the cochain complex $\mathcal{C}^*(\mathcal{E}nd_A)$ associated to the endomorphism operad of A. □

Corollary 2.9. [7, p. 65] *Let $\mathcal{H}$ be a bialgebra. Then $Cotor^*_{\mathcal{H}}(\mathbb{k}, \mathbb{k})$ is a Gerstenhaber algebra.*

Proof. Denote by μ and 1 the multiplication and the unit of $\mathcal{H}$. Denote by Δ and ε the diagonal and the counit of $\mathcal{H}$. For each $n \in \mathbb{N}$, denote by $\Delta^{n-1} : \mathcal{H} \to \mathcal{H}^{\otimes n}$ the $(n-1)$ iterated diagonal defined by $\Delta^{-1} := \varepsilon$, $\Delta^0 := Id_{\mathcal{H}}$ and $\Delta^{n+1} := (\Delta \otimes id_{\mathcal{H}}^{\otimes n}) \circ \Delta^n$. For an element $a \in \mathcal{H}$, we denote $\Delta^{n-1} a := a^{(1)} \otimes \cdots \otimes a^{(n)}$ or simply $a^1 \otimes \cdots \otimes a^n$. Here the sum is implicit and contrarily to Sweedler notation, we use upperscripts instead of lowerscripts, since we will need indices but no powers.

Consider the operad with multiplication O defined by $O(n) := \mathcal{H}^{\otimes n}$ and if $a_1 \otimes \cdots \otimes a_m \in \mathcal{H}^{\otimes m}$ and $b_1 \otimes \cdots \otimes b_n \in \mathcal{H}^{\otimes n}$,

$$(a_1 \otimes \cdots \otimes a_m) \circ_i (b_1 \otimes \cdots \otimes b_n) :=$$
$$a_1 \otimes \cdots \otimes a_{i-1} \otimes (\Delta^{n-1} a_i) \cdot (b_1 \otimes \cdots \otimes b_n) \otimes a_{i+1} \otimes \cdots \otimes a_m =$$
$$a_1 \otimes \cdots \otimes a_{i-1} \otimes a_i^1 b_1 \otimes \cdots \otimes a_i^n b_n \otimes a_{i+1} \otimes \cdots \otimes a_m.$$

Here $\cdot$ denotes the product on the tensor product of algebras, $\mathcal{H}^{\otimes n}$. The identity element id of O is $1 \in \mathcal{H}^{\otimes 1}$. The multiplication μ is $1 \otimes 1 \in \mathcal{H}^{\otimes 2}$. The element e of O is the unit of $\mathbb{k}$, $1_{\mathbb{k}} \in \mathcal{H}^{\otimes 0}$. The cochain complex associated to this operad is the Cobar construction on $\mathcal{H}$, denoted usually $\Omega\mathcal{H}$. Since $\mathrm{Cotor}^*_{\mathcal{H}}(\mathbb{k}, \mathbb{k}) = H^*(\Omega\mathcal{H})$, the result follows from 1.3. □

If $\mathcal{H}$ is cocommutative, this operad is symmetric and coincides with the semi-direct product $\underline{Ass} \rtimes \mathcal{H}$ as defined by Salvatore and Wahl [15]. This operad is the dual of the cooperad considered by van der Laan [19, 4.10 Lemma].

3. Cyclic operads with multiplication

3.1. A *Batalin-Vilkovisky algebra* is a Gerstenhaber algebra G equipped with a degree -1 linear map $B : G^i \to G^{i-1}$ such that $B \circ B = 0$ and

$$\{a, b\} = (-1)^{|a|} \Big(B(a \cup b) - (Ba) \cup b - (-1)^{|a|} a \cup (Bb) \Big) \tag{3.2}$$

for a and $b \in G$.

Definition 3.3. A *cyclic operad* is a non-Σ-operad O equipped with linear maps $\tau_n : O(n) \to O(n)$ for $n \in \mathbb{N}$ such that

$$\forall n \in \mathbb{N}, \quad \tau_n^{n+1} = id_{O(n)}, \tag{3.4}$$

$$\forall m \geq 1, n \geq 1, \quad \tau_{m+n-1}(f \circ_1 g) = \tau_n g \circ_n \tau_m f, \tag{3.5}$$

$$\forall m \geq 2, n \geq 0, 2 \leq i \leq m, \quad \tau_{m+n-1}(f \circ_i g) = \tau_m f \circ_{i-1} g, \tag{3.6}$$

for each $f \in O(m)$ and $g \in O(n)$. In particular, we have $\tau_1 id = id$.

This definition is taken out of [12, p. 247-8] except that since our operad O is not necessarily symmetric, we don't assume that the action of the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$ on $O(n)$ extends to an action of the symmetric group of order $n+1$, S_{n+1}.

Remark that (3.5) and (3.6) are equivalent to

$$\forall m \geq 1, n \geq 1, \quad \tau_{m+n-1}^{-1}(f \circ_m g) = \tau_n^{-1} g \circ_1 \tau_m^{-1} f, \tag{3.7}$$

$$\forall m \geq 2, n \geq 0, 1 \leq i \leq m-1, \quad \tau_{m+n-1}^{-1}(f \circ_i g) = \tau_m^{-1} f \circ_{i+1} g, \tag{3.8}$$

If instead of (3.5) and (3.6), τ_n satisfies

$$\tau_{m+n-1}(f \circ_m g) = \tau_n g \circ_1 \tau_m f,$$
$$\tau_{m+n-1}(f \circ_i g) = \tau_m f \circ_{i+1} g,$$

as in the original definition of cyclic operad of Getzler and Kapranov [9, (2.2)], replace τ_n by τ_n^{-1}.

We will use the following generalizations of (3.8) and of (3.7): For each $m \geq 1$, $n \geq 0$, $1 \leq i \leq m$ and $j \in \mathbb{Z}$,

$$\text{if } 1 \leq i+j \leq m \text{ then } \tau_{m+n-1}^{-j}(f \circ_i g) = \tau_m^{-j} f \circ_{i+j} g, \tag{3.9}$$

$$\text{if } m+1 \leq i+j \leq m+n \text{ then } \tau_{m+n-1}^{-j}(f \circ_i g) = \tau_n^{-j+m-i} g \circ_{i+j-m} \tau_m^{i-m-1} f. \tag{3.10}$$

Definition 3.11. A *cyclic operad with multiplication* is an operad which is both an operad with multiplication and a cyclic operad such that

$$\tau_2 \mu = \mu.$$

The operad $\underline{Ass}$ is a cyclic operad: the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$ acts trivially on $\underline{Ass}(n) := \mathbb{k}$. A cyclic operad O is an cyclic operad with multiplication if and only if O is equipped with a morphism of cyclic operads $\underline{Ass} \to O$.

Definition 3.12. [11, 6.1.1] A *cocyclic module* C_n is a cosimplicial module endowed for all $n \geq 0$ with an action of the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$ on C_n subject to the following relations

$$\tau_n \delta_i = \delta_{i-1} \tau_{n-1} \text{ and } \tau_n \sigma_i = \sigma_{i-1} \tau_{n+1} \text{ for } 1 \leq i \leq n.$$

Proof of part a) of Theorem 1.4. Let $f \in O(n-1)$. By (3.5) and $\tau_2 \mu = \mu$

$$\tau_n \delta_1 f = \tau_n(f \circ_1 \mu) = \tau_2 \mu \circ_2 \tau_{n-1} f = \delta_0 \tau_{n-1} f.$$

By (3.6), for $2 \leq i \leq n-1$

$$\tau_n \delta_i f = \tau_n(f \circ_i \mu) = \tau_n f \circ_{i-1} \mu = \delta_{i-1} \tau_{n-1} f.$$

By (3.5),

$$\tau_n \delta_n f = \tau_n(\mu \circ_1 f) = \tau_{n-1} f \circ n - 1 \mu = \delta_{n-1} \tau_{n-1} f.$$

Let $g \in O(n+1)$. By (3.6), for $1 \leq j \leq n$,

$$\tau_n \sigma_j g = \tau_n(g \circ_{j+1} e) = \tau_{n+1} g \circ_j e = \sigma_{j-1} \tau_{n+1} g.$$

Therefore the cosimplicial module O is in fact a cocyclic module. □

4. Hochschild cohomology of a symmetric algebra

The cyclic endomorphism operad [12]**.** Let V be a module equipped with a bilinear form $\varphi : V \otimes V \to \mathbb{k}$ such that the associated right linear map $\Theta : V \xrightarrow{\cong} V^\vee$, $v \mapsto \varphi(-, v)$, is an isomorphism (i. e. φ is a non-degenerate bilinear form if V is a finite dimensional vector space). Consider the adjonction map

$$Ad : \mathrm{Hom}(V^{\otimes n}, V^\vee) \xrightarrow{\cong} \mathrm{Hom}(V^{\otimes n+1}, \mathbb{k}) \tag{4.1}$$

which associates to any $g \in \mathrm{Hom}(V^{\otimes n}, V^\vee)$, the map

$$Ad(g) : V^{\otimes n+1} \to \mathbb{k}, \quad v_0, v_1, \cdots, v_n \mapsto g(v_1, \cdots, v_n)(v_0).$$

The composite

$$\mathrm{Hom}(V^{\otimes n}, V) \overset{\mathrm{Hom}(V^{\otimes n},\Theta)}{\longrightarrow} \mathrm{Hom}(V^{\otimes n}, V^\vee) \overset{Ad}{\to} \mathrm{Hom}(V^{\otimes n+1}, \mathbb{k})$$

is an isomorphism. Explicitly this composite sends $f \in \mathrm{Hom}(V^{\otimes n}, V)$ to the linear map $\widehat{f} : V^{\otimes n+1} \to \mathbb{k}$ defined by

$$\widehat{f}(v_0, v_1, \cdots, v_n) = \varphi(v_0, f(v_1, \cdots, v_n)) \quad \text{for } v_0, v_1, \cdots, v_n \in V.$$

The cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$ acts on $V^{\otimes n+1}$ by permutation of factors:

$$t_n(v_0, \cdots, v_n) := (v_n, v_0, \cdots, v_{n-1}) \quad \text{for } (v_0, \cdots, v_n) \in V^{\otimes n+1}. \tag{4.2}$$

Define $\tau_n := t_n^\vee : \mathrm{Hom}(V^{\otimes n+1}, \mathbb{k}) \to \mathrm{Hom}(V^{\otimes n+1}, \mathbb{k})$. Using the identification $f \mapsto \widehat{f}$, we define $\tau_n : \mathrm{Hom}(V^{\otimes n}, V) \to \mathrm{Hom}(V^{\otimes n}, V)$ by $\widehat{\tau_n f} := \tau_n \widehat{f}$ for $f \in \mathrm{Hom}(V^{\otimes n}, V)$. Explicitly, $\tau_n(f)$ is the unique map such that

$$\varphi(v_0, \tau_n f(v_1, \cdots, v_n)) = \varphi(v_n, f(v_0, \cdots, v_{n-1})) \quad \text{for } v_0, \cdots, v_n \in V.$$

The endomorphism operad of V, $\mathcal{E}nd_V$, equipped with this last linear map $\tau_n : \mathcal{E}nd_V(n) \to \mathcal{E}nd_V(n)$ is a cyclic operad if and only if the bilinear form φ is symmetric.

Hochschild (co)homology. Let A be an algebra. Let M be an A-bimodule. Denote by $\mathcal{C}^*(A, M)$ the Hochschild cochain complex of A with coefficient in M [11, 1.5.1] and by $\mathcal{C}_*(A, M)$ the Hochschild chain complex [11, 1.1.1]. Recall that $\mathcal{C}^n(A, M) := \mathrm{Hom}(A^{\otimes n}, M)$ and that $\mathcal{C}_n(A, M) := M \otimes A^{\otimes n}$.

Consider an algebra A that is symmetric in the sense of representation theory [5]. By definition, it means that the algebra A is equipped with an isomorphism $\Theta : A \overset{\cong}{\to} A^\vee$ of A-bimodules. By functoriality, $\mathcal{C}^*(A, \Theta) : \mathcal{C}^*(A, A) \overset{\cong}{\to} \mathcal{C}^*(A, A^\vee)$ is an isomorphism of cosimplicial modules. The adjunction map (4.1) $Ad : \mathcal{C}^*(A, A^\vee) \overset{\cong}{\to} \mathcal{C}_*(A, A)^\vee$ is an isomorphism of cosimplicial modules (Compare with [11, 1.5.5]). Let $t_n : \mathcal{C}_n(A, A) \to \mathcal{C}_n(A, A)$ be the cyclic operator defined by 4.2. The Hochschild chain complex $\mathcal{C}_*(A, A)$ is a cyclic module [11, 2.1.0]. So $\mathcal{C}_*(A, A)^\vee$ with $\tau_n := t_n^\vee$ is a cocyclic module. Therefore by isomorphism, $\mathcal{C}^*(A, A)$ is also a cocyclic module.

Theorem 1.6 claims that this cocyclic structure on $\mathcal{C}^*(A, A)$ defines a structure of Batalin-Vilkovisky on the Gerstenhaber algebra $HH^*(A, A)$.

Proof of Theorem 1.6. Let $\varphi : A \otimes A \to \mathbb{k}$ be a bilinear form on A. It is easy to see that the associated right linear map $\Theta : A \to A^\vee$ is a morphism of A-bimodules if and only if φ is symmetric and

$$\varphi(a_2, a_0 a_1) = \varphi(a_0, a_1 a_2), \quad \forall a_0, a_1, a_2 \in A. \tag{4.3}$$

Therefore the endomorphism operad of the symmetric algebra A, $\mathcal{E}nd_A$ is cyclic: it is the cyclic endomorphism operad defined above. By definition, $\tau_2\mu$ is the unique map $A\otimes A\to A$ such that

$$\varphi(a_0,\tau_2(\mu)(a_1,a_2))=\varphi(a_2,a_0a_1),\quad \forall a_0,a_1,a_2\in A.$$

Therefore, by (4.3), we have $\tau_2\mu=\mu$. In the proof of Corollary 2.8, we have seen that $\mathcal{E}nd_A$ is an operad with multiplication. Therefore, $\mathcal{E}nd_A$ is a cyclic operad with multiplication and by Theorem 1.4, $HH^*(A,A)$ is a Batalin-Vilkovisky algebra. □

5. Cyclic cohomology of Hopf algebras

Let $\mathcal{H}$ be a Hopf algebra with antipode S and unity $\eta:\mathbb{k}\to\mathcal{H}$, $\eta(1_{\mathbb{k}})=1$. Consider a morphism of algebras (called *character*) $\chi:\mathcal{H}\to\mathbb{k}$. The *twisted antipode* $\widetilde{S}$ is by definition the convolution product of $\eta\circ\chi$ and S in $\mathrm{Hom}(\mathcal{H},\mathcal{H})$. Explicitly, for $h\in\mathcal{H}$, $\widetilde{S}(h)=\chi(h^1)S(h^2)$, where $\Delta h=h^1\otimes h^2$.

5.1. The couple $(\chi,1)$ is called a *modular pair in involution* for the Hopf algebra $\mathcal{H}$ if $\widetilde{S}\circ\widetilde{S}=id_{\mathcal{H}}$.

The twisted antipode $\widetilde{S}$ is an algebra antihomomorphism:

$$\widetilde{S}(ab)=\widetilde{S}(b)\widetilde{S}(a),\quad \forall a,b\in\mathcal{H},\quad \widetilde{S}(1)=1.$$

It is also a coalgebra twisted antihomomorphism:

$$\Delta\widetilde{S}(h)=S(h^2)\otimes\widetilde{S}(h^1),\quad \forall h\in\mathcal{H}.$$

More generally, we have

$$\forall n\geq 1,\quad \Delta\widetilde{S}(h)=S(h^n)\otimes\cdots\otimes S(h^2)\otimes\widetilde{S}(h^1). \tag{5.2}$$

Consider the map $\tau_n:\mathcal{H}^{\otimes n}\to\mathcal{H}^{\otimes n}$ defined by

$$\begin{aligned}\tau_n(h_1\otimes\cdots\otimes h_n) &:= \left(\Delta^{n-1}\widetilde{S}(h_1)\right)\cdot(h_2\otimes\cdots\otimes h_n\otimes 1)\\ &= \widetilde{S}(h_1)^1h_2\otimes\cdots\widetilde{S}(h_1)^{n-1}h_n\otimes\widetilde{S}(h_1)^n.\end{aligned}$$

Here $\cdot$ is the product in $\mathcal{H}^{\otimes n}$ and $\Delta^{n-1}\widetilde{S}(h_1)=\widetilde{S}(h_1)^1\otimes\cdots\otimes\widetilde{S}(h_1)^n$ (Review the notation introduced in the proof of Corollary 2.9).

In [3, 4], Connes and Moscovici have shown that the Cobar construction on $\mathcal{H}$ equipped with the maps τ_n is a cocyclic module if $(\chi,1)$ is a modular pair in involution.

Proof of Theorem 1.1. In the proof of Corollary 2.9, we have seen that $\Omega\mathcal{H}$ is an operad with multiplication. In order to apply Theorem 1.4, we need to see that $\Omega\mathcal{H}$ is a cyclic operad with multiplication. Therefore it remains to prove (3.5) and (3.6) and that $\tau_2\mu=\mu$.

Proof of (3.5). Let $(a_1,\cdots,a_m)\in\mathcal{H}^{\otimes m}$ and $(b_1,\cdots,b_n)\in\mathcal{H}^{\otimes n}$. Since $\widetilde{S}$ is an algebra antihomomorphism and Δ^{m+n-2} is an algebra morphism,

$$\Delta^{m+n-2}\widetilde{S}(a_1^1b_1)=\Delta^{m+n-2}\widetilde{S}(b_1)\cdot\Delta^{m+n-2}\widetilde{S}(a_1^1).$$

So $\tau_{m+n-1}\left[(a_1,\cdots,a_m)\circ_1(b_1,\cdots,b_n)\right]$
$= \Delta^{m+n-2}\widetilde{S}(b_1)\cdot\Delta^{m+n-2}\widetilde{S}(a_1^1)\cdot(a_1^2,\cdots,a_1^n,1,\cdots,1)\cdot(b_2,\cdots,b_n,a_2,\cdots,a_m,1).$
Since Δ is coassociative,

$$\left(\widetilde{S}(b_1)^{(1)},\cdots,\widetilde{S}(b_1)^{(n-1)},\widetilde{S}(b_1)^{(n)1},\cdots,\widetilde{S}(b_1)^{(n)m}\right)=\Delta^{m+n-2}\widetilde{S}(b_1).$$

So $\tau_n(b_1,\cdots,b_n)\circ_n\tau_m(a_1,\cdots,a_m)$
$= \Delta^{m+n-2}\widetilde{S}(b_1)\cdot(1,\cdots,1,\widetilde{S}(a_1)^1,\cdots,\widetilde{S}(a_1)^m)\cdot(b_2,\cdots,b_n,a_2,\cdots,a_m,1).$
Therefore to prove (3.5), it suffices to prove that
(5.3)

$$\Delta^{m+n-2}\widetilde{S}(a_1^1)\cdot(a_1^2,\cdots,a_1^n,1,\cdots,1)=(1,\cdots,1,\widetilde{S}(a_1)^1,\cdots,\widetilde{S}(a_1)^m).$$

Since $\widetilde{S}$ is a twisted antihomomorphism of coalgebras (5.2),

$$\begin{aligned}(5.4)\quad &\Delta^{m+n-2}\widetilde{S}(a_1^1)\cdot(a_1^2,\cdots,a_1^n,1,\cdots,1)\\ &=(S\left(a_1^{(1)m+n-1}\right)a_1^2,S\left(a_1^{(1)m+n-2}\right)a_1^3,\cdots,S\left(a_1^{(1)m+1}\right)a_1^n,\\ &\qquad S\left(a_1^{(1)m}\right),\cdots,S\left(a_1^{(1)2}\right),\widetilde{S}\left(a_1^{(1)1}\right)).\end{aligned}$$

We prove (5.3) by induction on $n\in\mathbb{N}^*$:

Case $n=1$. Since $a_1^1=\Delta^0a_1=a_1$, the two terms of (5.3) are equal to $\Delta^{m-1}\widetilde{S}(a_1)$.

Case $n\geq 2$. Suppose that (5.3) is true for $n-1$.

$$\Delta^{m+n-2}\widetilde{S}(a_1^1)\cdot(a_1^2,\cdots,a_1^n,1,\cdots,1)$$

using (5.4), since S is an antipode

$$\begin{aligned}&=(\varepsilon\left(a_1^{(1)m+n-1}\right)1,S\left(a_1^{(1)m+n-2}\right)a_1^2,\cdots,S\left(a_1^{(1)m+1}\right)a_1^{n-1},\\ &\qquad S\left(a_1^{(1)m}\right),\cdots,S\left(a_1^{(1)2}\right),\widetilde{S}\left(a_1^{(1)1}\right))\\ &=(1,S\left(\varepsilon\left(a_1^{(1)m+n-1}\right)a_1^{(1)m+n-2}\right)a_1^2,\cdots,S\left(a_1^{(1)m+1}\right)a_1^{n-1},\\ &\qquad S\left(a_1^{(1)m}\right),\cdots,S\left(a_1^{(1)2}\right),\widetilde{S}\left(a_1^{(1)1}\right))\end{aligned}$$

since ε is a counit

$$\begin{aligned}&=(1,S\left(a_1^{(1)m+n-2}\right)a_1^2,\cdots,S\left(a_1^{(1)m+1}\right)a_1^{n-1},\\ &\qquad S\left(a_1^{(1)m}\right),\cdots,S\left(a_1^{(1)2}\right),\widetilde{S}\left(a_1^{(1)1}\right))\end{aligned}$$

using (5.4) with n replaced by $n-1$,

$$=(1,\Delta^{m+n-3}\widetilde{S}(a_1^1)\cdot(a_1^2,\cdots,a_1^{n-1},1,\cdots,1))$$

by induction hypothesis

$$=(1,1,\cdots,1,\widetilde{S}(a_1)^1,\cdots,\widetilde{S}(a_1)^m).$$

Proof of (3.6). Since Δ^{n-1} is a morphism of algebras,

$$\Delta^{n-1}(\widetilde{S}(a_1)^{i-1}a_i) = \Delta^{n-1}(\widetilde{S}(a_1)^{i-1}) \cdot \Delta^{n-1}(a_i).$$

So $\tau_m(a_1, \cdots, a_m) \circ_{i-1} (b_1, \cdots, b_n)$

$$= (\widetilde{S}(a_1)^1 a_2, \cdots, \widetilde{S}(a_1)^{i-2} a_{i-1}, \Delta^{n-1}(\widetilde{S}(a_1)^{i-1}) \cdot \Delta^{n-1}(a_i) \cdot (b_1, \cdots, b_n), \widetilde{S}(a_1)^i a_{i+1}, \cdots, \widetilde{S}(a_1)^{m-1} a_m, \widetilde{S}(a_1)^m).$$

Since Δ is coassociative (case $n \geq 1$) and counitary (case $n = 0$),

$$\widetilde{S}(a_1)^1 \otimes \cdots \otimes \widetilde{S}(a_1)^{i-2} \otimes \Delta^{n-1}\widetilde{S}(a_1)^{i-1} \otimes \widetilde{S}(a_1)^i \otimes \cdots \otimes \widetilde{S}(a_1)^m = \Delta^{m+n-2}\widetilde{S}(a_1).$$

Therefore $\tau_m(a_1, \cdots, a_m) \circ_{i-1} (b_1, \cdots, b_n)$

$$= \Delta^{m+n-2}\widetilde{S}(a_1) \cdot (a_2, \cdots, a_{i-1}, \Delta^{n-1}(a_i) \cdot (b_1, \cdots, b_n), a_{i+1}, \cdots, a_m, 1) = \tau_{m+n-1}\left((a_1, \cdots, a_m) \circ_i (b_1, \cdots, b_n)\right).$$

The multiplication μ on the operad $\Omega\mathcal{H}$ is $1 \otimes 1$. Since $\widetilde{S}(1) = 1$, it is easy to check that $\tau_2\mu = \mu$. □

6. Proof of part b) of Theorem 1.4

Denote by B the Connes coboundary map associated to the cocyclic module O. By 1.3, we already know that $H(\mathcal{C}^*(O))$ is a Gerstenhaber algebra. Therefore to prove part b) of Theorem 1.4, it suffices to prove that (3.2) holds in cohomology.

Normalization. We would like to use the normalized cochain complex instead of the unnormalized one, since the formula for Connes coboundary map B is simpler in the normalized cochain complex. By definition, the normalized cochain complex associated to O, denoted $\overline{\mathcal{C}}^*(O)$, is the subcomplex of $\mathcal{C}^*(O)$ defined by

$$\overline{\mathcal{C}}^n(O) := \{f \in \mathcal{C}^n(O) \text{ such that } \sigma_j f = 0 \text{ for } 0 \leq j \leq n-1\}.$$

It is well known that the inclusion $\overline{\mathcal{C}}^*(O) \overset{\simeq}{\hookrightarrow} \mathcal{C}^*(O)$ is a cochain homotopy equivalence. It is easy to see that if $f \in \overline{\mathcal{C}}^m(O)$ and $g \in \overline{\mathcal{C}}^n(O)$ then $f \cup g \in \overline{\mathcal{C}}^{m+n}(O)$ and $f \circ_i g \in \overline{\mathcal{C}}^{m+n-1}(O)$ for $1 \leq i \leq m$. Therefore $\overline{\mathcal{C}}^*(O)$ is both a subalgebra and a sub Lie algebra of $\mathcal{C}^*(O)$. And so, it suffices to show that for any cycles f and $g \in \overline{\mathcal{C}}^*(O)$, (3.2) holds modulo coboundaries.

Reduction. In this section, we show that in order to prove (3.2), it suffices to prove Proposition 1.9. The idea behind that reduction is to start by proving the following particular case of (3.2): If $f \in H(\mathcal{C}^*(O))$ is of even degree then $B(f \cup f)$ is divisible by 2 and

$$f \bar{\circ} f = \frac{1}{2}\{f, f\} = \frac{1}{2}B(f \cup f) - (Bf) \cup f.$$

Remark that the number of terms in this formula is half the number of terms appearing in (3.2). Proposition 1.9 is a slight generalization of this formula. Lemma 1.8 implies in particular that $B(f \cup f)$ is a multiple of 2 if f is of even degree.

The bilinear map of degree -1

$$Z : \overline{\mathcal{C}}^m(O) \otimes \overline{\mathcal{C}}^n(O) \to \overline{\mathcal{C}}^{m+n-1}(O), \quad f \otimes g \mapsto Z(f,g)$$

is defined by

$$Z(f,g) := (-1)^{mn} \sum_{j=1}^{m} (-1)^{j(m+n-1)} \tau_{m+n-1}^{-j} \sigma_{m+n}(g \cup f). \tag{6.1}$$

Here $\sigma_n : O(n) \to O(n-1)$ is the extra degeneracy operator defined by $\sigma_n := \sigma_{n-1}\tau_n$.

In order to prove Lemma 1.8, we need the following two equations

$$\sigma_{m+n}(f \cup g) = \tau_m f \circ_m g. \tag{6.2}$$

Proof. By (3.6), (3.5) and since $\tau_2 \mu = \mu$,

$$\begin{aligned}\tau_{m+n}(f \cup g) = \tau_{m+n}((\mu \circ_1 f) \circ_{m+1} g) = \tau_{m+1}(\mu \circ_1 f) \circ_m g \\ = (\tau_m f \circ_m \tau_2 \mu) \circ_m g = (\tau_m f \circ_m \mu) \circ_m g.\end{aligned}$$

Therefore since $\mu \circ_2 e = id$

$$\sigma_{m+n}(f \cup g) = [(\tau_m f \circ_m \mu) \circ_m g] \circ_{m+n} e = \tau_m f \circ_m g.$$

□

$$\tau_{m+n-1}^{-n} \sigma_{m+n}(f \cup g) = \sigma_{m+n}(g \cup f). \tag{6.3}$$

Proof. Using (3.10) and equation (6.2),

$$\tau_{m+n-1}^{-n} \sigma_{m+n}(f \cup g) = \tau_{m+n-1}^{-n}(\tau_m f \circ_m g) = \tau_n^{-n} g \circ_n f = \sigma_{m+n}(g \cup f)$$

□

Proof of Lemma 1.8. The operator $N : O(n-1) \to O(n-1)$ is defined [4, (2.17)] by

$$N := \sum_{i=0}^{n-1} (-1)^{i(n-1)} \tau_{n-1}^{i} = \sum_{j=1}^{n} (-1)^{j(n-1)} \tau_{n-1}^{-j}.$$

By definition [4, (2.21)], Connes normalized cochain coboundary is $B := N\sigma_n : O(n) \to O(n-1)$. Therefore, using equation (6.3),

$$\begin{aligned}
B(f\cup g) &= \sum_{j=1}^{n}(-1)^{j(m+n-1)}\tau_{m+n-1}^{-j}\sigma_{m+n}(f\cup g)\\
&+\sum_{j=n+1}^{m+n}(-1)^{j(m+n-1)}\tau_{m+n-1}^{-(j-n)}\tau_{m+n-1}^{-n}\sigma_{m+n}(f\cup g)\\
&= \sum_{j=1}^{n}(-1)^{j(m+n-1)}\tau_{m+n-1}^{-j}\sigma_{m+n}(f\cup g)\\
&+\sum_{j=1}^{m}(-1)^{(j+n)(m+n-1)}\tau_{m+n-1}^{-j}\sigma_{m+n}(g\cup f)\\
&= (-1)^{mn}Z(g,f)+Z(f,g)
\end{aligned}$$

□

6.4. Let $f\in\overline{\mathcal{C}}^m(O)$ and $g\in\overline{\mathcal{C}}^n(O)$. Define for any $1\le j\le p\le m-1$

$$H_{j,p}(f,g) := (-1)^{jm-j+(n-1)(p+1+m)}\tau_{m+n-2}^{-j}\sigma_{m+n-1}(f\circ_{p-j+1} g).$$

and consider the bilinear map of degree -2,

$$H:\overline{\mathcal{C}}^m(O)\otimes\overline{\mathcal{C}}^n(O)\to\overline{\mathcal{C}}^{m+n-2}(O),$$

$$f\otimes g\mapsto H(f,g) := \sum_{1\le j\le p\le m-1} H_{j,p}(f,g).$$

Proof of part b) of Theorem 1.4 assuming Proposition 1.9. By applying Proposition 1.9 and Lemma 1.8,

$$\begin{aligned}
&f\bar{\circ}g+\varepsilon g\bar{\circ}f+dH(f,g)+\varepsilon dH(g,f)\\
&+H(df,g)+\varepsilon H(dg,f)+(-1)^{m-1}H(f,dg)+\varepsilon(-1)^{n-1}H(g,df)\\
&=(-1)^m Z(f,g)+\varepsilon(-1)^n Z(g,f)-(-1)^m(Bf)\cup g-\varepsilon(-1)^n(Bg)\cup f\\
&=(-1)^m\left(B(f\cup g)-(Bf)\cup g-(-1)^{m(n-1)}(Bg)\cup f\right).
\end{aligned}$$

Here the sign ε is equal to $-(-1)^{(m-1)(n-1)}=(-1)^{mn+m+n}$. Since in cohomology, the cup product is graded commutative, relation (3.2) is proved. □

Proof of Proposition 1.9. Recall that since $f\in O(m)$,

$$df=\mu\circ_2 f+\sum_{i=1}^{m}(-1)^i f\circ_i\mu+(-1)^{m+1}\mu\circ_1 f.$$

It is easy to see that Proposition 1.9 is a consequence of the following six equations.

$$(6.5)\qquad (-1)^m Z(f,g)-f\bar{\circ}g=H(\mu\circ_2 f,g)+H\left((-1)^{m+1}\mu\circ_1 f,g\right).$$

$$\sum_{1\leq j<p\leq m} H_{j,p}((-1)^{p-j} f\circ_{p-j}\mu, g) = (-1)^m H(f,\mu\circ_2 g). \tag{6.6}$$

(6.7)
$$\sum_{1\leq j\leq p\leq m-1} H_{j,p}\left((-1)^{p-j+1} f\circ_{p-j+1}\mu, g\right) = (-1)^m H\left(f,(-1)^{n+1}\mu\circ_1 g\right).$$

$$\sum_{1\leq j\leq m} H_{j,m}\left((-1)^{m-j+1} f\circ_{m-j+1}\mu, g\right) = -(-1)^m (Bf)\cup g. \tag{6.8}$$

$$\begin{aligned}\sum_{1\leq j\leq p\leq m} H_{j,p}(&\sum_{\substack{1\leq i\leq m,\\ i\neq p-j, i\neq p-j+1}} (-1)^i f\circ_i \mu, g)\\ &= -\mu\circ_2 H(f,g) - (-1)^{m+n-1}\mu\circ_1 H(f,g)\\ &\quad - \sum_{1\leq j\leq p\leq m-1}\ \sum_{\substack{1\leq i\leq p-1 \text{ or}\\ p+n\leq i\leq m+n-2}} (-1)^i H_{j,p}(f,g)\circ_i\mu.\end{aligned} \tag{6.9}$$

$$\sum_{\substack{1\leq j\leq p\leq m-1\\ p\leq i\leq p+n-1}} (-1)^i H_{j,p}(f,g)\circ_i\mu = (-1)^m H\left(f,\sum_{i=1}^{n}(-1)^i g\circ_i\mu\right). \tag{6.10}$$

Proof of (6.5). By separating the terms $j=p$ and $j<p$,

$$\begin{aligned}H(\mu\circ_2 f,g) &= \sum_{1\leq j\leq p\leq m} (-1)^{jm+(n-1)(p+m)}\tau_{m+n-1}^{-j}\sigma_{m+n}\left((\mu\circ_2 f)\circ_{p-j+1} g\right)\\ &= (-1)^m Z(f,g)\\ &\quad + \sum_{1\leq j<p\leq m} (-1)^{jm+(n-1)(p+m)}\tau_{m+n-1}^{-j}\sigma_{m+n}(id\cup f\circ_{p-j} g).\end{aligned}$$

On the other hand, since (6.3) $\tau_{m+n-1}^{-1}\sigma_{m+n}(f\circ_{p-j+1} g\cup id) = \sigma_{m+n}(id\cup f\circ_{p-j+1} g)$,

$$\begin{aligned}&(-1)^{m+1}H(\mu\circ_1 f,g)\\ &= \sum_{1\leq j\leq p\leq m} (-1)^{m+1+jm+(n-1)(p+m)}\tau_{m+n-1}^{-(j-1)}\sigma_{m+n}(id\cup f\circ_{p-j+1} g).\end{aligned}$$

Therefore, since (6.2) $\sigma_{m+n}(id\cup f\circ_p g) = \tau_1 id\circ_1 (f\circ_p g) = f\circ_p g$, by the change of variables $j'=j-1$,

$$\begin{aligned}(-1)^{m+1}H(\mu\circ_1 f,g) &= -f\bar{\circ}g\\ &\quad - \sum_{1\leq j'<p\leq m} (-1)^{j'm+(n-1)(p+m)}\tau_{m+n-1}^{-j'}\sigma_{m+n}(id\cup f\circ_{p-j'} g).\end{aligned}$$

□

Proof of (6.6). By the change of variables $p' = p - 1$,

$$\sum_{1\leq j<p\leq m} H_{j,p}\left((-1)^{p-j} f \circ_{p-j} \mu, g\right)$$
$$= \sum_{1\leq j\leq p'\leq m-1} (-1)^{p'+1-j+jm+(n-1)(p'+1+m)} \tau_{m+n-1}^{-j}\sigma_{m+n}\left(f \circ_{p'+1-j} (\mu \circ_2 g)\right)$$
$$= (-1)^m H(f, \mu \circ_2 g).$$

□

The proof of (6.7) is similar.

To prove the last three equations, we will express all the formulas in terms of composite $\circ_i$ of the elements $\tau_m^{-j} f$, g, μ and e, using again and again (3.10) and (3.9). Therefore, we start by giving a new expression for $H_{j,p}(f, g)$:

$$H_{j,p}(f, g) = (-1)^{jm-j+(n-1)(p+1+m)}\sigma_{j-1}(\tau_m^{-j} f \circ_{p+1} g). \tag{6.11}$$

Proof. We have seen that O is a cocyclic module. Therefore [1, Remark 1.2], the following relation between τ_n and the degeneracy maps σ_i holds

$$\forall\, 0 \leq r \leq i \leq n, \quad \tau_n^r\sigma_i = \sigma_{i-r}\tau_{n+1}^r.$$

For the extra degeneracy map σ_{n+1}, we have

$$\forall\, 0 \leq r \leq n, \quad \tau_n^r\sigma_{n+1} = \sigma_{n-r}\tau_{n+1}^{r+1}$$

or equivalently

$$\forall\, 1 \leq j \leq n+1, \quad \tau_n^{-j}\sigma_{n+1} = \sigma_{j-1}\tau_{n+1}^{-j}. \tag{6.12}$$

Therefore using (3.9),

$$\tau_{m+n-2}^{-j}\sigma_{m+n-1}(f \circ_{p-j+1} g) = \sigma_{j-1}\tau_{m+n-1}^{-j}(f \circ_{p-j+1} g) = \sigma_{j-1}(\tau^{-j} f \circ_{p+1} g).$$

□

Proof of (6.8). By (6.12),

$$B(f) = \sum_{j=1}^{m}(-1)^{j(m-1)}\sigma_{j-1}\tau_m^{-j} f.$$

By (6.11) and (3.10),

$$\sum_{1\leq j\leq m} H_{j,m}\left((-1)^{m-j+1} f \circ_{m-j+1} \mu, g\right)$$
$$= \sum_{j=1}^{m}(-1)^{m+1+jm+j}\sigma_{j-1}\left[(\tau_2^{-1}\mu \circ_1 \tau_m^{-j} f) \circ_{m+1} g\right]$$
$$= -(-1)^m (Bf) \cup g.$$

□

Proof of (6.9). In all this proof, we put $\varepsilon := (-1)^{i+jm+(n-1)(p+m)}$. By (6.11),

$$\sum_{1\leq j\leq p\leq m} H_{j,p}\Big(\sum_{\substack{1\leq i\leq m,\\ i\neq p-j, i\neq p-j+1}} (-1)^i f\circ_i \mu, g\Big)$$
$$=\sum_{\substack{1\leq j\leq p\leq m, 1\leq i\leq m,\\ i+j<p \text{ or } i+j>p+1}} \varepsilon\sigma_{j-1}\left[\tau_{m+1}^{-j}(f\circ_i\mu)\circ_{p+1} g\right].$$

Remark that when $m+1\leq i+j$, we can forget the condition $i+j\neq p$ under theses sums, and that when $m+2\leq i+j$, we can also forget the condition $i+j\neq p+1$.

Using respectively (3.9), (3.10), (3.10) again and (3.10) twice, we obtain that

$$\tau_{m+1}^{-j}(f\circ_i\mu)=\begin{cases}\tau_m^{-j}f\circ_{i+j}\mu & \text{if } i+j\leq m,\\ \mu\circ_1\tau_m^{-j}f & \text{if } i+j=m+1,\\ \mu\circ_2\tau_m^{-(j-1)}f & \text{if } i+j=m+2,\\ \tau_m^{-(j-1)}f\circ_{i+j-m-2}\mu & \text{if } m+3\leq i+j.\end{cases}$$

By the change of variables $i'=i+j$, we have $\varepsilon=(-1)^{i'-j+jm+(n-1)(p+m)}$,

$$\sum_{\substack{1\leq j\leq p\leq m,\\ 1\leq i<p-j}} \varepsilon\sigma_{j-1}\left[(\tau_m^{-j}f\circ_{i+j}\mu)\circ_{p+1} g\right]$$
$$=\sum_{1\leq j<i'<p\leq m}\varepsilon\gamma(\tau_m^{-j}f;id,\dots,id,e,id,\dots,id,\mu,id,\dots,id,g,id,\dots,id)$$
$$=-\sum_{1\leq j\leq i<p\leq m-1}(-1)^i H_{j,p}(f,g)\circ_i\mu,$$

where in the second sum, e is the j-th element after the semi-colon, μ is the i'-th element and g is the p-th element. And we have

$$\sum_{\substack{1\leq j\leq p\leq m,\\ p-j+1<i\leq m-j}} \varepsilon\sigma_{j-1}\left[(\tau_m^{-j}f\circ_{i+j}\mu)\circ_{p+1} g\right]$$
$$=\sum_{\substack{1\leq j\leq p\leq m,\\ p+1<i'\leq m}}\varepsilon\gamma(\tau_m^{-j}f;id,\dots,id,e,id,\dots,id,g,id,\dots,id,\mu,id,\dots,id)$$
$$=-\sum_{\substack{1\leq j\leq p\leq m-1,\\ p+n\leq i\leq m+n-2}}(-1)^i H_{j,p}(f,g)\circ_i\mu,$$

where in the second sum, e is the j-th element after the semi-colon, g is the $(p+1)$-th element and μ is the i'-th element.

For $k=1$ or $k=2$,

$$\mu\circ_k H(f,g)=\sum_{1\leq j\leq p\leq m-1}(-1)^{jm-j+(n-1)(p+1+m)}\mu\circ_k\left[(\tau_m^{-j}f\circ_{p+1} g)\circ_j e\right].$$

Since when $1 \leq j \leq p \leq m$ and $i+j=m+1$, we have $1 \leq i \leq m$ and the equivalence

$$i+j \neq p+1 \Longleftrightarrow p \neq m,$$

$$\sum_{\substack{1\leq j\leq p\leq m, 1\leq i\leq m,\\ i+j=m+1, i+j\neq p+1}} \varepsilon\sigma_{j-1}\left[(\mu \circ_1 \tau_m^{-j} f) \circ_{p+1} g\right] = -(-1)^{m+n-1}\mu \circ_1 H(f,g).$$

Since when $1 \leq j \leq p \leq m$ and $i+j=m+2$, we have the equivalence

$$1 \leq i \leq m \Longleftrightarrow 2 \leq j,$$

by the change of variables $j' = j-1$ and $p' = p-1$,

$$\sum_{\substack{1\leq j\leq p\leq m, 1\leq i\leq m,\\ i+j=m+2}} \varepsilon\sigma_{j-1}\left[(\mu \circ_2 \tau_m^{-(j-1)} f) \circ_{p+1} g\right] = -\mu \circ_2 H(f,g).$$

By the change of variables $j' = j-1$, $i' = i+j-m-2$ and then $p' = p-1$, we have $\varepsilon = (-1)^{i'-j'-1+j'm+(n-1)(p+m)}$ and

$$\sum_{\substack{1\leq j\leq p\leq m,\\ m+3-j\leq i\leq m}} \varepsilon\sigma_{j-1}\left[(\tau_m^{-(j-1)} f \circ_{i+j-m-2} \mu) \circ_{p+1} g\right]$$

$$= \sum_{1\leq i'<j'<p\leq m} \varepsilon\gamma(\tau_m^{-j'} f; id,\ldots,id,\mu,id,\ldots,id,e,id,\ldots,id,g,id,\ldots,id)$$

$$= -\sum_{1\leq i'<j'\leq p'\leq m-1} (-1)^{i'} H_{j',p'}(f,g) \circ_{i'} \mu,$$

where in the second sum, μ is the i'-th element after the semi-colon, e is the j'-th element and g the p-th element. □

To prove (6.10), use (6.11) and the change of variables $i' = i-p+1$.

7. Proof of Corollary 1.5

By definition, the negative cyclic cohomology of a cocyclic module is the cyclic homology of its associated mixed cochain complex. Using the following Proposition, we see immediately that Corollary 1.5 follows from Theorem 1.4.

In this section, all the graded modules are considered as lower graded. Recall that a *mixed complex* is a graded module $M = \{M_i\}_{i\in\mathbb{Z}}$ equipped with a linear map of degree -1 $d : M_i \to M_{i-1}$ and a linear map of degree $+1$ $B : M_i \to M_{i+1}$ such that $d^2 = B^2 = dB + Bd = 0$.

Proposition 7.1. *Let (M, d, B) be a mixed complex such that its homology $H_*(M, d)$ equipped with $H_*(B)$ has a Batalin-Vilkovisky algebra structure. Then its cyclic homology $HC_*(M)$ is a graded Lie algebra of lower degree $+2$.*

The key point in the proof of this proposition is the following lemma not explicited stated in [2]. The proof of this lemma is exactly the proof of Theorem 6.1 of [2].

Lemma 7.2. *Let H be a Batalin-Vilkovisky algebra and HC be a graded module. Consider a long exact sequence of the form*

$$\cdots \to H_n \xrightarrow{I} HC_n \to HC_{n-2} \xrightarrow{\partial} H_{n-1} \to \cdots$$

If the operator $B : H_i \to H_{i+1}$ is equal to $\partial \circ I$ then

$$[a, b] := (-1)^{|a|} I\left(\partial a \cup \partial b\right), \quad \forall a, b \in HC$$

defines a Lie bracket of degree +2 on HC.

Proof of Proposition 7.1. A mixed complex M is a (differential graded) module over the differential exterior graded algebra $\Lambda := (\Lambda\varepsilon_1, 0)$ [10]. Consider the Bar construction of Λ with coefficients in M, $B(M; \Lambda; \Bbbk)$. By [10, Proposition 1.4], the cyclic homology of M, is the homology of $B(M; \Lambda; \Bbbk)$:

$$HC_*(M) := H_*(B(M; \Lambda; \Bbbk)) = \mathrm{Tor}_*^{\Lambda}(M, \Bbbk).$$

Explicitly, $B(M; \Lambda; \Bbbk)$ is the complex defined as follow:

$$B(M; \Lambda; \Bbbk)_n = M_n \oplus M_{n-2} \oplus M_{n-4} \oplus \cdots \quad \text{and}$$
$$d(m_n, m_{n-2}, m_{n-4}, \cdots) = (dm_n + Bm_{n-2}, dm_{n-2} + Bm_{n-4}, \cdots).$$

A mixed complex yields a Connes long exact sequence

$$\cdots \to H_n(M, d) \xrightarrow{I} HC_n(M) \xrightarrow{S} HC_{n-2}(M) \xrightarrow{\partial} H_{n-1}(M, d) \to \cdots$$

(Usually ∂ is unfortunately denoted by B, since it is induced by B.) The connecting homomorphism $\partial : H_{n-2}(B(M, \Lambda, \Bbbk)) \to H_{n-1}(M, d)$ maps the class of the $n-2$ cycle $(m_{n-2}, m_{n-4}, \cdots)$ to the class of the cycle Bm_{n-2} [16, Proof of Prop 2.3.6]. Of course, $I : H_n(M, d) \to H_n(B(M, \Lambda, \Bbbk))$ maps the class of the cycle m_n to $(m_n, 0, 0, \cdots)$. Therefore $H_*(B) = \partial \circ I$. Finally by Lemma 7.2, $HC_*(M)$ is a graded Lie algebra of degree +2. □

8. Comparison with McClure and Smith

In this section, we compare Theorem 1.4 with a result announced by McClure and Smith in [14]:

A cosimplicial module (resp. space) $X^\bullet$ has a *cup-cocyclic* structure if it is a cocyclic module (resp. space) and if it has a cup product [13, Definition 2.1(iii)] such that

$$\tau_{m+n+1}^{-n-1}(f \cup \delta_0 g) = g \cup \delta_0 f, \quad \forall f \in X^m, g \in X^n.$$

8.1. [14, Remark 15.10] *If the cosimplicial module $X^\bullet$ has a cup-cocyclic structure then the normalized cochain complex associated, $\overline{\mathcal{C}}^*(X^\bullet)$, has an action by an operad equivalent to the singular chains on the operad $\mathcal{F}$ of framed little disks.*

So $H^*(\overline{\mathcal{C}}^*(X^\bullet))$ has an action by the operad $H_*(\mathcal{F})$, i. e. $H^*(\overline{\mathcal{C}}^*(X^\bullet))$ is a Batalin-Vilkovisky algebra. Tedious computations show that a cosimplicial module (resp. space) has a cup-cocyclic structure if and only if it is a linear (resp. topological) cyclic operad with multiplication in our sense. Therefore their result gives a Deligne version of our Theorem.

Note that McClure and Smith have announced a topological counterpart to their result:

8.2. [14, Theorem 14.5] *If the cosimplicial space $X^\bullet$ has a cup-cocyclic structure then its realisation, $Tot(X^\bullet)$, has an action by an operad equivalent to the operad $\mathcal{F}$ of framed little disks.*

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UMR 6093 associée au CNRS, Université d'Angers, Faculté des Sciences, 2 Boulevard Lavoisier, 49045 Angers, FRANCE
E-mail address: firstname.lastname@univ-angers.fr